A simple proof for the multinomial version of Representation Theorem

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Abstract

In this work we present a demonstration for the multinomial version of de Finetti's Representation Theorem. We use characteristic functions, following his first demonstration for binary random quantities, but simplify the argument through forward operators.

Keywords: characteristic function; exchangeability; forward operator.

1 Introduction

In 1928, Bruno de Finetti presented a contribution at the International Congress of Mathematicians: *Funzione caratteristica di un fenomeno aleatorio*, but it was published only in 1932, when the sixth and last volume of the annals of that congress was released. In 1930, a more detailed version was published, as a *memoir*, by the *Accademia dei Lincei*, with the same title.[1]

De Finetti used an analytic argument and characteristic functions, alongside with the exchangeability hypothesis, to prove his famous Representation Theorem. In this work we use his arguments with forward operators to present a new proof for the multinomial case.

2 De Finetti's method for multinomial trials

Let an infinite sequence of random quantities, that assume any of k values or categories, considered to be exchangeable. We want to study some subsequence

[1] See [5] and [6] and [1].

of n of such quantities. In order to do this, we consider the vector $(S_1, \ldots, S_{k-1})$ that displays the number of quantities that assumed value $1, 2, \ldots, k-1$. The probability generating function (p.g.f) of such vector is, for $z \in \mathbb{C}^k$:

$$\Omega_n(z_1, \ldots, z_{k-1}) = \sum_{\Delta} \omega^{(n)}_{h_1,\ldots,h_{k-1}} z_1^{h_1} \ldots z_{k-1}^{h_{k-1}} \tag{1}$$

in which $\Delta = \{(x_1, \ldots, x_{k-1}) \in \mathbb{Z}_+^{k-1} : x_1 + \ldots x_{k-1} \leq n\}$, $n \in \mathbb{N}$ and $\omega^{(n)}_{h_1,\ldots,h_{k-1}}$ is the probability that we observe, in n trials, h_1 of category 1, ..., h_{k-1} of category $k-1$, regardless of the order. We also denote by $\omega_0^{(0)} = 1$, $t = \sum_{i=1}^{k-1} h_i$ and

$$\omega^{(t)}_{h_1,\ldots,h_{k-1}} = \omega_{h_1,\ldots,h_{k-1}}.$$

Exchangeability makes it possible to write

$$\frac{\omega^{(n)}_{h_1,\ldots,h_{k-1}}}{\binom{n}{h_1,\ldots,h_{k-1}}} = \delta^{n-t}\, \omega_{h_1,\ldots,h_{k-1}} \geq 0. \tag{2}$$

in which the δ operator is defined after [3] by

$$\delta\omega_{h_1,\ldots,h_{k-1}} = \omega_{h_1,\ldots,h_{k-1}} - \omega_{h_1+1,\ldots,h_{k-1}} - \omega_{h_1,h_2+1,\ldots,h_{k-1}} - \ldots - \omega_{h_1,\ldots,h_{k-1}+1}.$$

Now we define the forward operators. Let us denote

$$F_j^k \omega_{h_1,\ldots,h_{k-1}} = \omega_{h_1,\ldots,h_j+k,\ldots,h_{k-1}}$$

then it follows that

$$F_j^r F_j^k \omega_{h_1,\ldots,h_{k-1}} = F_j^k F_i^r \omega_{h_1,\ldots,h_{k-1}} = \omega_{h_1,\ldots,h_i+r,\ldots,h_j+k,\ldots,h_{k-1}}$$

and that

$$\delta^{n-t}\omega_{h_1,\ldots,h_{k-1}} = (1 - F_1 - \ldots - F_{k-1})^{n-t}\omega_{h_1,\ldots,h_{k-1}}.$$

Using (2), implied by the exchangeability hypothesis, and forward operators, the p.g.f (1) may be written as:

$$\begin{aligned}
\Omega_n(z_1, \ldots, z_{k-1}) &= \sum_{\Delta} \omega^{(n)}_{h_1,\ldots,h_{k-1}} z_1^{h_1} \ldots z_{k-1}^{h_{k-1}} = \\
&= \sum_{\Delta} \binom{n}{h_1, \ldots, h_{k-1}} z_1^{h_1} \ldots z_{k-1}^{h_{k-1}} (1 - F_1 \ldots - F_{k-1})^{n-h_1 \ldots - h_{k-1}} \omega_{h_1,\ldots,h_{k-1}} \\
&= \sum_{\Delta} \binom{n}{h_1, \ldots, h_{k-1}} (z_1 F_1)^{h_1} \ldots (z_{k-1}F_{k-1})^{h_{k-1}} (1 - F_1 \ldots - F_{k-1})^{n-s} \omega_0 \\
&= [1 + F_1(z_1 - 1) + \ldots + F_{k-1}(z_{k-1} - 1)]^n \omega_0.
\end{aligned} \tag{3}$$

Following de Finetti's approach, we define the characteristic function of $\overline{S} = (S_1/n, \ldots, S_{k-1}/n)$ which, using (3), may be written as:

$$\begin{aligned}
\Psi_{\overline{S}}(t_1, \ldots, t_{k-1}) &= \Omega_n(e^{it_1/n}, \ldots, e^{it_{k-1}/n}) \\
&= [1 + F_1(e^{it_1/n} - 1) + \ldots + F_{k-1}(e^{it_{k-1}/n} - 1)]^n \omega_0.
\end{aligned}$$

and study its limit when $n \to \infty$. It is possible to show[2] that:

$$\Psi_\Theta(t_1, \ldots, t_{k-1}) = \exp[i(F_1 t_1 + F_2 t_2 + \ldots + F_{k-1} t_{k-1})]\omega_0. \tag{4}$$

and, by Levy's continuity theorem, it is know that (4) is the characteristic function of only one random vector, Θ, that assumes value in the $k-1$ simplex and whose distribution function[3] is denoted Φ. Given the properties relating moments and characteristic functions, we can rewrite (1) once more, through the multinomial theorem:

$$\Omega_n(z_1, \ldots, z_{k-1}) = \int_{\mathbb{S}^{k-1}} [1 + \theta_1(z_1 - 1) + \ldots + \theta_{k-1}(z_{k-1} - 1)]^n d\Phi(\theta) =$$
$$= \sum_{\Delta} \binom{n}{h_1, \ldots, h_{k-1}} z_1^{h_1} \ldots z_{k-1}^{h_{k-1}} \int_{\mathbb{S}^{k-1}} \theta_1^{h_1} \theta_2^{h_2} \ldots (1 - \theta_1 - \ldots - \theta_{k-1})^{n-t} d\Phi(\theta)$$

where $\theta \in \mathbb{S}^{k-1}$, the $(k-1)$-simplex, and from it follows that

$$\omega^{(n)}_{h_1,\ldots,h_{k-1}} = \binom{n}{h_1, \ldots, h_{k-1}} \int_{\mathbb{S}^{k-1}} \theta_1^{h_1} \theta_2^{h_2} \ldots (1 - \theta_1 - \ldots - \theta_{k-1})^{n-t} d\Phi(\theta)$$

that is de Finetti's Representation Theorem for multinomial sequences of exchangeable random quantities.

De Finetti [7] does not provide a proof for the multinomial case but only asymptotical arguments that, starting from the finite binomial case, it is possible to derive the infinite multinomial case. For the binomial case, [2] provides a proof based on [8], but for the for the multinomial case the proof is reported as "a straightforward, albeit algebraic cumbersome, generalization of the proof of [Representation theorem for binary random quantities]". The proof given by [8] can be considered as, essentially, de Finetti's proof with a limit argument not involving characteristic functions.

The results presented here provided a simple and clean demonstration of de Finetti's Representation Theorem for infinite sequences of multinomial random quantities.

[2] The limits with forward operators are well defined because the set of polynomial operators induces an algebra that is isomorphic to the algebra of polynomials in real or complex variables. See [4].

[3] It is possible to find the distribution function inverting the characteristic function.

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