Poisson Driven Stationary Markovian Models

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Abstract

In this paper we concisely summarize some recent findings that can be found in [5]. We propose a simple but yet powerful method to construct strictly stationary Markovian models with given but arbitrary invariant distributions. The idea is based on a Poisson transform modulating the dependence structure in the model. An appealing feature of our approach is that we are able to fully control the underlying transition probabilities and therefore incorporate them within standard estimation methods. We analyze some specific cases in both discrete and continuous time, and in particular focus on models with invariant distributions belonging to the gamma family or well-known transformations of it. Given the representation of the transition density, a Gibbs sampler algorithm, based on the slice method, is proposed and implemented. The resulting methodology is of particular interest for the estimation of certain continuous time models, such as diffusion processes.

Keywords: Markov process; Slice method; Stationary model.

1 Markovian Model

In this paper we sketch results that are extensively presented and proved in [5] about the construction of Poisson driven stationary Markovian models. In particular, stationary processes represent the crucial component in several modeling approaches used in probability and statistics, given quite simple estimation and prediction procedures. It is worth noticing that if it is possible to relax the distributional assumptions of the stationary distribution, then many of the drawbacks of stationary models are substantially reduced. When thinking of random phenomena evolving in time a natural starting point is to consider Markovian processes and thus look for transition mechanisms that retain a particular invariant distribution over time. This is the approach followed by many of the constructions available in the literature, in both discrete and continuous time. Many of the available approaches start from a stochastic equation describing the dynamics in time. Unfortunately, the availability of analytic expressions for the corresponding transition probabilities is not always immediate. However, a full control of the transition probabilities driving a Markovian process is desirable, especially for considerable advantages in estimation and prediction procedures.

In [11] the reversibility property characterizing Gibbs sampler Markov chains is exploited to propose strictly stationary AR(1)-type models with virtually any choice of marginal distribution. In particular, they demonstrate that the approach by [4] can be seen as a particular case. Being such a general approach, concrete choices of dependence should be made to accommodate specific modeling needs. Indeed, specific instances of this construction, meeting some specific dependency or some distributional flexibility, can be found in [6], [7, 8, 9] and [3]. Of particular interest is the approach to continuous time Markovian models studied in [10].

In this work we construct stationary Markovian models using a Poisson transform. Although our proposal results in a particular dependence structure, this is general enough and allows to construct models with prescribed invariant distributions supported on $\mathbb{R}_+$, which after simple transformations can be extended to processes with other state-spaces.

2 Poisson weighted density

Let f a continuous density function supported on $\mathbb{R}_+$. For any $y \in \{0, 1, 2, \ldots\}$ and $\phi > 0$, we define the *Poisson weighted density* as

$$\hat{f}(x; y, \phi) := \frac{x^y e^{-x\phi} f(x)}{\xi(y, \phi)} \quad \text{where} \quad \xi(y, \phi) := \int_{\mathbb{R}_+} z^y e^{-z\phi} f(z) \mathrm{d}z \tag{1}$$

Notice that (1) constitutes a well defined probability density on $\mathbb{R}_+$, provided the above integral exists. For $\phi \downarrow 0$, it reduces to the size-biased density of f and, when $y = 0$, it reduces to the Esscher transform of f. Density (1) can also be seen as the posterior density of a Poisson $\mathsf{Po}(\phi x)$ distribution with prior f on x. For constructing a stationary Markovian process, $(X_n)_{n\in\mathbb{Z}_+}$, with invariant distribution having density f we can use the Poisson weighted density, defining the time-homogeneous one-step ahead Markovian density

$$\begin{aligned} p(x_{n-1}, x_n) \quad &:= \quad \sum_{y=0}^{\infty} \hat{f}(x_n; y, \phi)\, \mathsf{Po}(y; x_{n-1}\phi) \\ &= \quad \exp\{-\phi(x_n + x_{n-1})\} f(x_n) \sum_{y=0}^{\infty} \frac{(x_n x_{n-1}\phi)^y}{y! \xi(y, \phi)} \end{aligned} \tag{2}$$

which clearly satisfies $p(x_{n-1}, x_n)f(x_{n-1}) = p(x_n, x_{n-1})f(x_n)$ for all $x_{n-1}, x_n \in \mathbb{R}_+$ leading to a time-reversible Markovian process.

After the description of the general construction based on a Poisson transform, we explore some particular cases of interest including constructions of diffusion models with gamma, generalized inverse gaussian and generalized extreme value marginal distributions.

2.1 Estimation

The availability of a tractable expression for the transition density is appealing in Markov processes analysis and estimation. In particular if the choice of f leads to a manageable analytic expression in (2), the maximum likelihood estimator (MLE) can be easily determined. Alternatively, if analytically perform the summation is forbidden in (2), one could make use of such a representation for the transition density and obtain a MLE via the expectation-maximization (EM) algorithm based on the augmented likelihood or a Gibbs sampler algorithm for Bayesian estimation. Some illustrations based on simulated and real data sets are also presented in [5].

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