Adversarial Risk Analysis

Fabrizio Ruggeri

Istituto di Matematica Applicata e Tecnologie Informatiche
Consiglio Nazionale delle Ricerche
Via Alfonso Corti 12, I-20133, Milano, Italy, European Union

fabrizio@mi.imati.cnr.it

www.mi.imati.cnr.it/fabrizio

ADVERSARIAL HYPOTHESIS TESTING (AHT)

- Use concepts from Adversarial Risk Analysis (ARA)
- Agent (Defender D) needs to ascertain which of several hypotheses holds, based on observations from a source
- Another agent (Attacker A) alters the observations to induce the Defender to make a wrong decision (and get a benefit)
- AHT problem studied from the Defender's perspective
- Defender needs to forecast the Attacker's decision, simulating from the corresponding Attacker's decision making problem

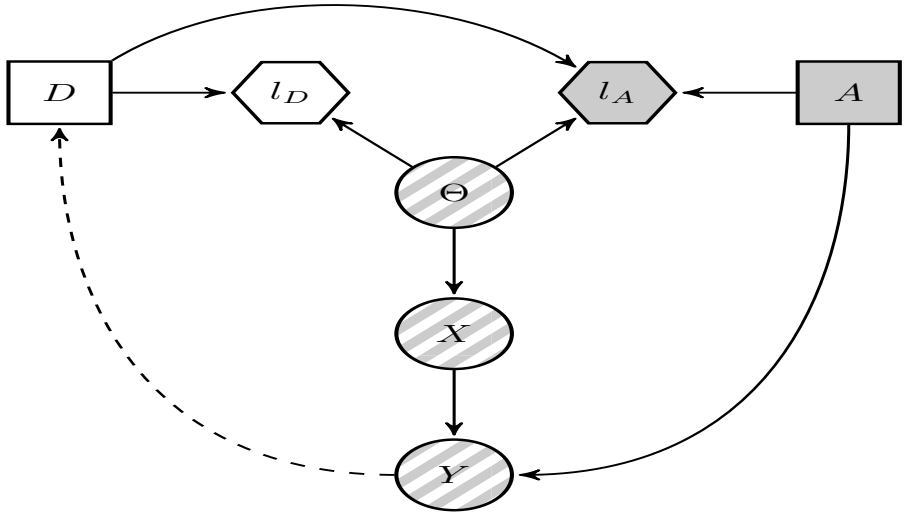
AHT: SIMPLE EXAMPLE

- Defender D needs to decide whether a batch of e-mails includes spam or not
- D has beliefs about the standard flow of legit and spam messages
- Attacker A alters such flow in an attempt to confound the Defender and gain some benefit
- Both agents obtain different rewards depending on whether
  - batch is accepted or not by the Defender
  - batch includes just legit messages or not

ADVERSARIAL HYPOTHESIS TESTING

- Test of two simple hypotheses: $\Theta = \{\theta_0, \theta_1\}$
- Observation x generated according to a model depending on θ
- x altered to y by A's action a
- y observed by D $\Rightarrow$ D's decision d on θ based on y, without observing x
- Depending on d and actual $\theta \Rightarrow$ losses (utilities) for both agents
- Efforts by A in minimising the loss
- Support for D in choosing θ to minimise the loss
- Example: $X \sim \mathcal{E}(\theta)$ failure time s.t. $\theta_0 \ll \theta_1 \Rightarrow$ better reliability for θ_0

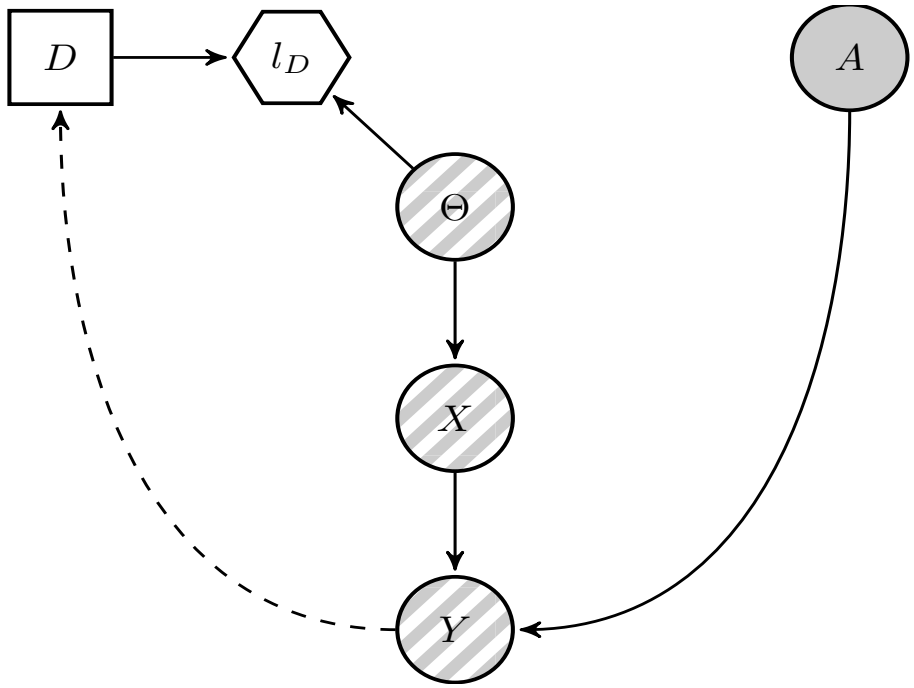
AHT: BI-AGENT INFLUENCE DIAGRAM (BAID)



- Decisions: D (depending on Y) and A
- Random: $\Theta \rightarrow X \rightarrow Y$ (Y influenced also by the decision A)
- Losses: l_D and l_A depending on Θ and related decisions (l_A also on decision D)
- Node: decision (square), uncertainty (circle), deterministic (double), utility (hex.)
- Arrow: conditional relation (solid), information available at decision time (dashed)

SOLVING THE DEFENDER'S PROBLEM

Influence diagram of the Defender's decision problem



Attacker's node is now random

SOLVING THE DEFENDER'S PROBLEM

Assessed by Defender D:

- Belief $\pi_D(\theta)$ on hypotheses:
  $p_D(\theta = \theta_i) = \pi_i^D, \quad i = 0, 1$, with $\pi_i^D \geq 0$ and $\pi_0^D + \pi_1^D = 1$
- Belief $\pi_D(x|\theta)$ on how data depend on the hypothesis:
  $X|\theta_i \sim \pi_D(x|\theta_i), \quad i = 0, 1$
- Belief $\pi_D(y|x, a)$ on how action $a \in \mathcal{A}$ by Attacker modifies actual x into observed y
- Belief $\pi_D(a)$ on the attack a performed by the Attacker
- Standard 0-1-c_D loss function $l_D(d, \theta)$ with decision space $\mathcal{D} = \{d_0, d_1\}$ s.t.
  $d_j = \{\text{Defender supports } \theta_j\}$, $j = 0, 1$

SOLVING THE DEFENDER'S PROBLEM

Defender's loss function

		Actual Hypothesis	
		θ_0	θ_1
D's Decision	d_0	0	1
	d_1	c_D	0

- 0 best loss, associated with the *right* decision
- $c_D \leq 1$ (without loss of generality)

SOLVING THE DEFENDER'S PROBLEM

- Solve: $\arg\min_{d\in\mathcal{D}} \sum_{i=0}^{1} l_D(d,\theta_i)\,\pi_D(\theta_i|y)$
- $\Rightarrow d_0$, i.e. support for θ_0, optimal solution for D if and only if $\pi_D(\theta_1|y) \le c_D\,\pi_D(\theta_0|y)$
- From

$$\begin{aligned}\pi_D(\theta_i|y) &= \frac{\pi_D(\theta_i,y)}{\pi_D(y)} = \frac{\iint \pi_D(\theta_i)\,\pi_D(y|x,a)\,\pi_D(x|\theta_i)\,\pi_D(a)\,\mathrm{d}x\,\mathrm{d}a}{\pi_D(y)}\\ &= \frac{\pi_i^D \iint \pi_D(y|x,a)\,\pi_D(x|\theta_i)\,\pi_D(a)\,\mathrm{d}x\,\mathrm{d}a}{\pi_D(y)}, \quad i=0,1\end{aligned}$$

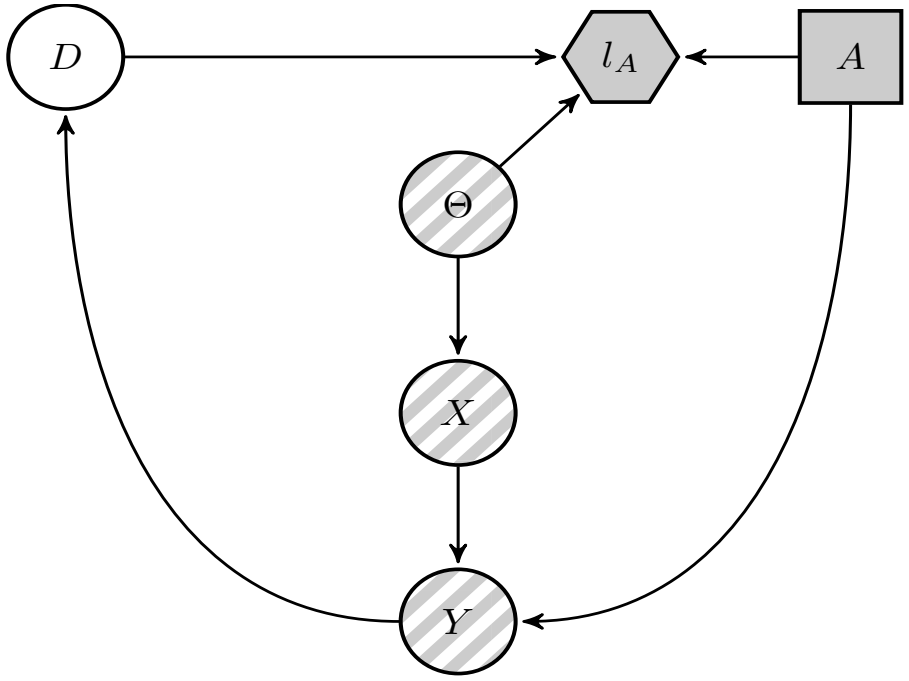
- $\Rightarrow$ support for θ_0, optimal decision for D if and only if

$$\pi_1^D \iint \pi_D(y|x,a)\,\pi_D(x|\theta_1)\,\pi_D(a)\,\mathrm{d}x\,\mathrm{d}a \le c_D\,\pi_0^D \iint \pi_D(y|x,a)\,\pi_D(x|\theta_0)\,\pi_D(a)\,\mathrm{d}x\,\mathrm{d}a$$

- Here we use Bayes Theorem $\pi_D(\theta_i|y) = \dfrac{\pi_D(y|\theta_i)\pi_i^D}{\pi_D(y)}$ and neglect the denominator in the minimisation

SOLVING THE ATTACKER'S PROBLEM

- All Defender's beliefs obtained in standard way, except for $\pi_D(a)$
- Defender's belief $\pi_D(a)$ on Attacker's action comes from considering his decision problem
- Defender's node is now random



SOLVING THE ATTACKER'S PROBLEM

Needed for Attacker A:

- Belief $\pi_A(\theta)$ on hypotheses:
  $p_A(\theta = \theta_i) = \pi_i^A, \quad i = 0, 1$, with $\pi_i^A \geq 0$ and $\pi_0^A + \pi_1^A = 1$
- Belief $\pi_A(x|\theta)$ on how data depend on the hypothesis:
  $X|\theta_i \sim \pi_A(x|\theta_i), \quad i = 0, 1$
- Belief $\pi_A(y|x, a)$ on consequences of his action $a \in \mathcal{A}$, modifying actual x into y
- Belief $\pi_A(d|y)$ on the decision d taken by the Defender upon observing y
- Loss function $l_A(d, \theta, a) = l_{jk}(a)$, with
  - $j = 0, 1$ depending on Defender's decision d_j (i.e., supporting θ_j)
  - $k = 0, 1$ depending on actual θ_k
  - No cost directly associated with chosen action a (but only on consequences)

SOLVING THE ATTACKER'S PROBLEM

Attacker's loss function

		Actual Hypothesis	
		θ_0	θ_1
D's Decision	d_0	$l_{00}(a)$	$l_{01}(a)$
	d_1	$l_{10}(a)$	$l_{11}(a)$

- Better for the Attacker if the Defender makes mistakes
  $\Rightarrow l_{00}(a) \geq l_{01}(a)$ and $l_{10}(a) \leq l_{11}(a)$

SOLVING THE ATTACKER'S PROBLEM

Attacker's loss function

		Actual Hypothesis	
		θ_0	θ_1
D's Decision	d_0	1	0
	d_1	c_A^1	c_A^2

$0 \leq c_A^1 \leq c_A^2 \leq 1$

- Best loss for Attacker (0) when Defender supports θ_0 and she should not
- Worst loss for Attacker (1) when Defender supports θ_0 and she should
- Intermediate cases: worse for Attacker when Defender supports θ_1 and actual hypothesis is θ_1

SOLVING THE ATTACKER'S PROBLEM

- Optimal decision for Attacker given by a^* s.t.

  $a^* = \arg\min_{a \in \mathcal{A}} \sum_{j=0}^{1} \sum_{i=0}^{1} \iint l_A(d_j, \theta_i, a)\, \pi_A(d_j|y)\, \pi_A(\theta_i)\, \pi_A(y|x,a)\, \pi_A(x|\theta_i)\, \mathrm{d}y\, \mathrm{d}x$

- Defender does not know $\pi_A(\theta)$, $\pi_A(x|\theta)$, $\pi_A(y|x,a)$, $\pi_A(d|y)$ and $l_A(d,\theta,a)$

- $\Rightarrow$ model uncertainty around them through random probabilities and losses
  $F = (\Pi_A(\theta), \Pi_A(x|\theta), \Pi_A(y|x,a), \Pi_A(d|y), L_A(d,\theta,a))$

- $\Rightarrow$ find optimal random attack

  $A^* = \arg\min_{a \in \mathcal{A}} \sum_{j=0}^{1} \sum_{i=0}^{1} \iint L_A(d_j, \theta_i, a)\, \Pi_A(d_j|y)\, \Pi_A(\theta_i)\, \Pi_A(y|x,a)\, \Pi_A(x|\theta_i)\, \mathrm{d}y\, \mathrm{d}x$

- $\Rightarrow$ required distribution through $\pi_D(a) = \Pi(A^* = a)$ (assuming discrete $\mathcal{A}$, but possible also for continuous one)

SOLVING THE ATTACKER'S PROBLEM

- $\pi_D(a)$ approximated through simulation, sampling from F
- Samples $\left(\Pi_A^k(\theta_i), \Pi_A^k(x|\theta_i), \Pi_A^k(y|x,a), \Pi_A^k(d_j|y), L_A^k(d_j,\theta_i,a)\right)$, $k = 1, ..., K$
- $\Rightarrow a_k^* = \arg\min_{a\in\mathcal{A}} \sum_{j=0}^{1} \sum_{i=0}^{1} \iint L_A^k(d_j,\theta_i,a)\, \Pi_A^k(d_j|y)\, \Pi_A^k(\theta_i)\, \Pi_A^k(y|x,a)\, \Pi_A^k(x|\theta_i)\, \mathrm{d}y\, \mathrm{d}x$
- $\Rightarrow \widehat{\pi}_D(a) \approx \#\{a_k^* = a\}/K$

SOLVING THE ATTACKER'S PROBLEM

Choice of random probabilities and loss F

- $\Pi_A(\theta)$ based on $\pi_D(\theta)$ with some uncertainty around it
  - $\Pi_A(\theta)$ modelled as a Dirichlet distribution with mean $\pi_D(\theta)$, if discrete
  - $\Pi_A(\theta)$ modelled as Dirichlet process with base measure $\pi_D(\theta)$, if continuous
- $\Pi_A(x|\theta)$ based on $\pi_D(x|\theta)$ with some uncertainty around it
- $\Pi_A(y|x,a)$ based on $\pi_D(y|x,a)$ with some uncertainty around it
- Parametric form for $L_A(d,\theta,a)$ with distribution over such parameters
- On the contrary, $\Pi_A(d|y)$ requires strategic thinking as the Defender needs to assess the Attacker's beliefs about which decision d she will make, given that she observes y
- $\Rightarrow$ could be the start of a hierarchy of decision making problems!

SOLVING THE ATTACKER'S PROBLEM

- Defender should solve the problem

  $\arg\min_{d\in\mathcal{D}} \sum_{i=0}^{1} l_D(d,\theta_i)\,\pi_D(\theta_i|y)$ equivalent to

  $\arg\min_{d\in\mathcal{D}} \sum_{i=0}^{1} \int\int l_D(d,\theta_i)\,\pi_D(\theta_i)\,\pi_D(y|x,a)\,\pi_D(x|\theta_i)\,\pi_D(a)\,\mathrm{d}x\,\mathrm{d}a$

- Attacker does not know ingredients of above integral

- $\Rightarrow$ assume uncertainty over them through random loss $L_D^A(d,\theta)$ and random distributions $\Pi_D^A(\theta)$, $\Pi_D^A(y|x,a)$, $\Pi_D^A(x|\theta)$ and $\Pi_D^A(a)$

- $\Rightarrow$ get corresponding random optimal decision

- Assessment of $\Pi_D^A(a)$ (what Defender believes that Attacker thinks about her beliefs concerning the attack to be implemented)
  $\Rightarrow$ strategic component leading to the next stage in the hierarchy

- Iterate until no further information is available, then choosing non-informative prior over the involved probabilities and losses

NUMERICAL EXAMPLE

- Two hypotheses: $\theta_0 = 2$ and $\theta_1 = 1$
- Two decisions: d_0 chooses $\theta_0 = 2$ and d_1 chooses $\theta_1 = 1$
- Priors over the hypotheses: $\pi_0^D = \pi_1^D = 1/2$
- Actual data $X|\theta_i$ exponentially distributed $\mathcal{E}(\theta_i)$, with uncertainty about θ_i
- Data x modified by Attacker into y, with actions
  - a_0: $x \to y = x$ (*keeping*)
  - a_1: $x \to y = 2x$ (*doubling*)
  - $a_{\text{-}1}$: $x \to y = x/2$ (*halving*)
- Suppose (for illustration) Defender knows probabilities $\pi_D(a)$ used by Attacker to choose actions:

  $\pi_D(a_0) = 1/2$, $\pi_D(a_1) = 1/6$ and $\pi_D(a_{-1}) = 1/3$

NUMERICAL EXAMPLE

- Two decisions: d_0 chooses $\theta_0 = 2$ and d_1 chooses $\theta_1 = 1$
- Loss function $L(d, \theta)$

		Actual Hypothesis	
		θ_0	θ_1
D's Decision	d_0	0	1
	d_1	3/4	0

NUMERICAL EXAMPLE

Adopt decision d_0 (i.e., accept $\theta_0 = 2$) if and only if

$$\pi_1^D \left[\theta_1\, e^{-\theta_1 y}\, \pi_D(a_0) + \theta_1\, e^{-\theta_1 \frac{y}{2}}\, \pi_D(a_1) + \theta_1\, e^{-\theta_1 2y}\, \pi_D(a_{-1})\right]$$
$$\leq$$
$$\tfrac{3}{4}\, \pi_0^D \left[\theta_0\, e^{-\theta_0 y}\, \pi_D(a_0) + \theta_0\, e^{-\theta_0 \frac{y}{2}}\, \pi_D(a_1) + \theta_0\, e^{-\theta_0 2y}\, \pi_D(a_{-1})\right]$$

- $\Leftrightarrow 2e^{-\frac{y}{2}} + 3e^{-y} - 5e^{-2y} - 6e^{-4y} \leq 0$
- $\Leftrightarrow y \lesssim 0.3723$ is observed

  (Note that $\theta = 2$ leads to a smaller mean w.r.t. $\theta = 1$, i.e. $1/2$ vs. 1)
- Note that a small change in probabilities, i.e. $\pi_0^D = 1/3$ and $\pi_1^D = 2/3$ (and other probabilities and losses kept as before) $\Rightarrow d_1$ optimal regardless of observed y

NUMERICAL EXAMPLE

Defender does not accurately know $\pi_D(a) \Rightarrow$ ARA

- $\Pi_A(\theta_1)$ drawn uniformly over $[1/4, 3/4]$, and $\Pi_A(\theta_0) = 1 - \Pi_A(\theta_1)$
- $\Pi_A(x|\theta)$, where $\theta \in \{\theta_0, \theta_1\}$, from a Gamma distribution $\mathcal{G}a(\alpha, \beta)$ with mean $\alpha/\beta = \theta$ and variance $\alpha/\beta^2 = \sigma^2$ uniformly chosen over $[1/2, 2]$ s.t. variance randomness induces that of $\Pi_A(x|\theta)$
- $\Pi_A(y|x, a)$ Dirac distributions coinciding with those of $\pi_D(y|x, a)$
- $\Pi_A(d|y)$ looking at the likelihood $h(y|d, a)$ of y under different choices of d and a, mixing them through a random allocation of probabilities to each action

NUMERICAL EXAMPLE

- Attacker assumes the Defender is modelling the data with an exponential distribution
- Likelihood $h(y|d,a)$ of y under different choices of d and a
  - d_0 chooses $\theta_0 = 2$ and d_1 chooses $\theta_1 = 1$
  - a_0 (*keeping*), a_1 (*doubling*) and a_{-1} (*halving*)
- Example
  - y reported and a_1 chosen $\Rightarrow x = y/2$ true value
  - d_0 chosen $\Rightarrow h(y|d_0,a_1) = 2e^{-y}$

		Actions		
		a_0	a_1	a_{-1}
D's Decision	d_0	$2e^{-2y}$	$2e^{-y}$	$2e^{-4y}$
	d_1	e^{-y}	$e^{-y/2}$	e^{-2y}

NUMERICAL EXAMPLE

- Defender assessing the probabilities $(\epsilon_0, \epsilon_1, \epsilon_{-1})$ assigned by the Attacker to each strategy through a Dirichlet distribution $\mathcal{D}ir(1,1,1)$

- $\Rightarrow$

$$P_A(d = d_1|\epsilon_0, \epsilon_1, \epsilon_{-1}, y) \qquad \frac{\sum_{j=-1}^{1} \epsilon_j \, h(y|d_1, a_j)}{\sum_{j=-1}^{1} \epsilon_j \, h(y|d_0, a_j) + \sum_{j=-1}^{1} \epsilon_j \, h(y|d_1, a_j)}$$

$$= \quad \frac{\epsilon_0 \, e^{-y} \epsilon_1 \, e^{-\frac{y}{2}} + \epsilon_{-1} \, e^{-2y}}{2\,(\epsilon_0 \, e^{-2y} + \epsilon_1 \, e^{-y} + \epsilon_{-1} \, e^{-4y}) + \epsilon_0 \, e^{-y} + \epsilon_1 \, e^{-\frac{y}{2}} + \epsilon_{-1} \, e^{-2y}}$$

- Distribution of $(\epsilon_0, \epsilon_1, \epsilon_{-1})$ induces the randomness of $P_A(d = d_1|y)$

- $P_A(d = d_0|y) = 1 - P_A(d = d_1|y)$

NUMERICAL EXAMPLE

Random loss function $L_A(d, \theta, a)$ based on table below

- C_A^1 fixed at 0
- C_A^2 uniformly drawn from $[1/2, 1]$

		Actual Hypothesis	
		θ_0	θ_1
D's Decision	d_0	1	0
	d_1	C_A^1	C_A^2

NUMERICAL EXAMPLE

- Attacker's random expected losses for the three actions
-
$$\Psi_A(a_0) = \int \left[\Pi_A(d_0|y=x)\,\Pi_A(\theta_0)\,\Pi_A(x|\theta_0) + C_A^2\,\Pi_A(d_1|y=x)\,\Pi_A(\theta_1)\,\Pi_A(x|\theta_1)\right] \mathrm{d}x$$
$$\Psi_A(a_1) = \int \left[\Pi_A(d_0|y=2x)\,\Pi_A(\theta_0)\,\Pi_A(x|\theta_0) + C_A^2\,\Pi_A(d_1|y=2x)\,\Pi_A(\theta_1)\,\Pi_A(x|\theta_1)\right] \mathrm{d}$$
$$\Psi_A(a_{\text{-}1}) = \int \left[\Pi_A(d_0|y=\tfrac{x}{2})\,\Pi_A(\theta_0)\,\Pi_A(x|\theta_0) + C_A^2\,\Pi_A(d_1|y=\tfrac{x}{2})\,\Pi_A(\theta_1)\,\Pi_A(x|\theta_1)\right] \mathrm{d}x$$
- Random models induce randomness in these expected losses
- $K = 100,000$ observations drawn from the corresponding distributions
- $\Rightarrow$ Estimates $\widehat{\pi}_D(a_0) \approx 0.04$, $\widehat{\pi}_D(a_1) \approx 0.85$ and $\widehat{\pi}_D(a_{\text{-}1}) \approx 0.11$
- Optimal action: d_0 when $y \lesssim 0.7374$ (different from previous solution)

NUMERICAL EXAMPLE

1 Set $p_j = 0, \quad \text{-}1 \leq j \leq 1.$

2 **For** $k = 1$ **to** K

3 Generate $\pi_A^{1,k} \sim \mathcal{U}(1/4, 3/4)$. Compute $\pi_A^{0,k} = 1 - \pi_A^{1,k}$.

4 Generate $\sigma_{0,k}^2 \sim \mathcal{U}(1/2, 2)$. Compute $\alpha_0^k = \theta_0^2/\sigma_{0,k}^2; \quad \beta_0^k = \theta_0/\sigma_{0,k}^2$.

5 Generate $\sigma_{1,k}^2 \sim \mathcal{U}(1/2, 2)$. Compute $\alpha_1^k = \theta_1^2/\sigma_{1,k}^2; \quad \beta_1^k = \theta_1/\sigma_{1,k}^2$

6 Generate $(\epsilon_0^k, \epsilon_1^k, \epsilon_{-1}^k) \sim \mathcal{D}ir(1, 1, 1)$..

7 Generate $C_A^{2,k} \sim \mathcal{U}(1/2, 1)$.

8
$$\psi_A^k(a_0) = \pi_A^{0,k} \int (1 - g(\epsilon_0, \epsilon_1, \epsilon_{-1}, x)) \; f(x|\alpha_0^k, \beta_0^k)\, \mathrm{d}x + C_A^{2,k}\, \pi_A^{1,k} \int g(\epsilon_0, \epsilon_1, \epsilon_{-1}, x)\, f(x|\alpha_1^k, \beta_1^k)\, \mathrm{d}x$$

9
$$\psi_A^k(a_1) = \pi_A^{0,k} \int (1 - g(\epsilon_0, \epsilon_1, \epsilon_{-1}, 2x)) \; f(x|\alpha_0^k, \beta_0^k)\, \mathrm{d}x + C_A^{2,k}\, \pi_A^{1,k} \int g(\epsilon_0, \epsilon_1, \epsilon_{-1}, 2x)\, f(x|\alpha_1^k, \beta_1^k)\, \mathrm{d}x$$

10
$$\psi_A^k(a_{-1}) = \pi_A^{0,k} \int (1 - g(\epsilon_0, \epsilon_1, \epsilon_{-1}, x/2)) \; f(x|\alpha_0^k, \beta_0^k)\, \mathrm{d}x + C_A^{2,k}\, \pi_A^{1,k} \int g(\epsilon_0, \epsilon_1, \epsilon_{-1}, x/2)\, f(x|\alpha_1^k, \beta_1^k)\, \mathrm{d}x$$

11 Determine $j^* = \underset{\text{-}1 \leq j \leq 1}{\arg\min}\, \psi_A^k(a_j)$.

12 Set $p_{j^*} = p_{j^*} + 1$.

13 Set $\hat{\pi}_D(a_j) = p_j/K, \quad \text{-}1 \leq j \leq 1.$

BATCH ACCEPTANCE MODEL

- Decision: accept or not a batch of items received over a period of time, some of which could be faulty, thus entailing potential security and/or performance problems
- Type of issues arising in areas such as screening containers at international ports, accepting batches of electronic messages or admitting packages of perishable products or electronic components, among others
- Consider different scenarios for a batch with m items in a period;
  - Loss depending if at least one faulty item is included (1 or m faulty items give the same loss)
  - Loss depending on the number of included faulty items among the m
- Consider different Attacker's strategies:
  - $\mathcal{S}_1$. Attacker adds some, new faulty items
  - $\mathcal{S}_2$. Attacker modifies few original items converting them into faulty ones
  - $\mathcal{S}_3$. Attacker combines strategies $\mathcal{S}_1$ and $\mathcal{S}_2$
- Start with non-adversarial hypothesis testing and then include adversaries

BATCH ACCEPTANCE MODEL

- Decision maker D (*Defender*) receives a batch with two types of items x
  - 0 (acceptable items)
  - 1 (faulty items)
- D needs to decide whether to accept (d_0) or reject (d_1) the batch
- D observes the batch size, modelled by a Poisson distribution $\mathcal{P}o(\lambda)$ over a unit period (or a homogeneous Poisson process, HPP, of parameter λ)
- Distribution on λ as a consequence of past experience:
  - Gamma prior $\mathcal{G}a(a, b)$ on λ
  - r items arrived after t periods $\Rightarrow$ posterior $\lambda|t, r \sim \mathcal{G}a(a + r, b + t)$
- λ will have no impact when D observes the actual value of m

BATCH ACCEPTANCE MODEL

- Item acceptable with probability θ

  Z designates item acceptability, s.t. $z = 0$ acceptable and $z = 1$ faulty

  $\Rightarrow p_D(z = 0|\theta) = \theta$ and $p_D(z = 1|\theta) = 1 - \theta$

- Acceptability of an item independent of the arrival process $\Rightarrow$ arrival of acceptable items is HPP of parameter $\lambda\theta$ (*Coloring or Thinning Theorem*)

- Early knowledge: Beta prior $\mathcal{B}e(\alpha, \beta)$ for θ

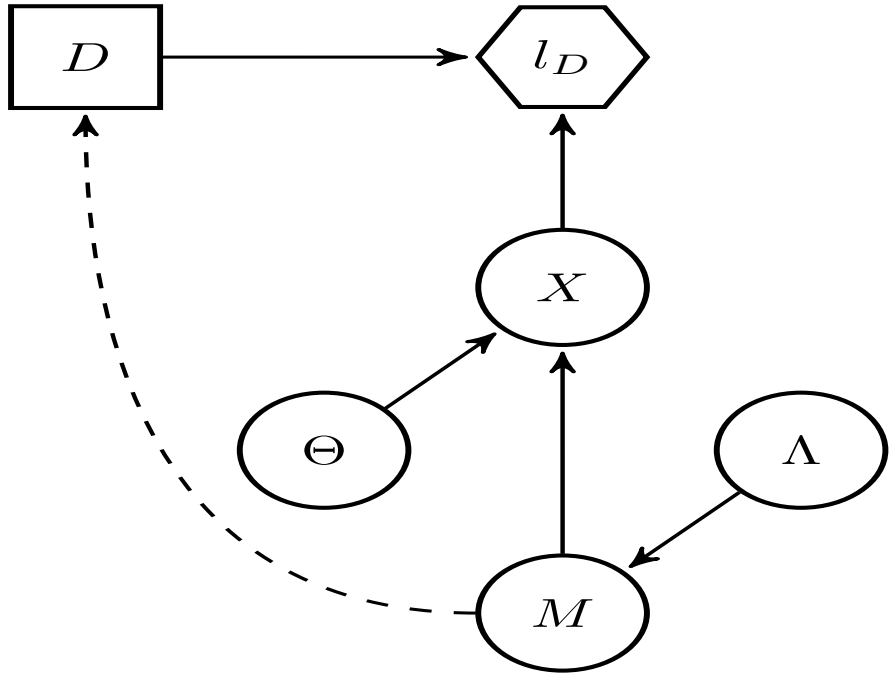
- Past observations: r received items with s acceptable (and $r - s$ faulty)

  $\Rightarrow$ posterior $\theta|r, s \sim \mathcal{B}e(\alpha + s, \beta + r - s)$

- To fix ideas, in a unit period we shall have
  - Total number of items $m|\lambda \sim \mathcal{P}o(\lambda)$
  - Total number of acceptable items $x|\lambda, \theta \sim \mathcal{P}o(\lambda\theta)$
  - (Conditional on m) total number of acceptable items $x|m, \theta \sim \mathcal{B}in(m, \theta)$

BATCH ACCEPTANCE MODEL

Influence diagram for batch acceptance problem without adversaries



BATCH ACCEPTANCE MODEL

Scenario A: Winner takes it all

- Batch with m items in a period
- Allowing one faulty item is as bad as allowing several of them, because of the entailed security or performance problems
- Loss function given by

		Batch of m Items		
		All Acceptable	Some Faulty	
		$p = \theta^m$	$p = 1 - \theta^m$	**Exp. Loss**
D's Decision	Accept, d_0	0	1	$1 - \theta^m$
	Reject, d_1	c	0	$c\,\theta^m$

BATCH ACCEPTANCE MODEL

- Suppose batch size m known to Defender $D \Rightarrow \lambda$ not relevant
- Expected losses of both decisions

$$l_D(d_0) = E_\theta\left[1-\theta^m\right] = 1 - E_\theta\left[\theta^m\right]$$
$$l_D(d_1) = E_\theta\left[c\,\theta^m\right] = c\,E_\theta\left[\theta^m\right]$$

- Decision: accept the batch (d_0) if and only if

$$1 - E_\theta\left[\theta^m\right] \leq c\,E_\theta\left[\theta^m\right] \quad \Longleftrightarrow \quad E_\theta\left[\theta^m\right] \geq \frac{1}{1+c}$$

- $E_\theta\left[\theta^m\right]$ decreases as m increases $\Rightarrow$ threshold value m_A
  $\Rightarrow$ rejection of the batch (d_1) if $m > m_A$
- m_A recursively obtained for posterior $\mathcal{B}e(\alpha+s, \beta+r-s)$ on θ from
  $E_\theta\left[\theta^m\right] = \prod_{k=0}^{m-1} \frac{\alpha+s+k}{\alpha+\beta+r+k}$

BATCH ACCEPTANCE MODEL

- Suppose batch size m unknown to Defender D, with distribution $p(m|\lambda), m \in \mathcal{N}$
- Expected losses of both decisions (now summing over all possible values of m)

$$l_D(d_0) = 1 - E_\theta \left(E_\lambda \left(\sum_{m=0}^{\infty} \theta^m p(m|\lambda) \right) \right)$$
$$l_D(d_1) = c\, E_\theta \left(E_\lambda \left(\sum_{m=0}^{\infty} \theta^m p(m|\lambda) \right) \right)$$

- Decision: accept the batch (d_0) if and only if

$$E_\theta \left(E_\lambda \left(\sum_{m=0}^{\infty} \theta^m p(m|\lambda) \right) \right) > \frac{1}{c+1}$$

- If $p(m|\lambda)$ Poisson, then accept the batch (d_0) if and only if

$$E_\theta \left(E_\lambda \left(e^{\lambda(\theta-1)} \right) \right) > \frac{1}{c+1}$$

BATCH ACCEPTANCE MODEL

- Gamma distribution $\mathcal{G}a(a,p)$ over λ and Beta distribution $\mathcal{B}e(\alpha,\beta)$ over θ
- $E_\lambda(e^{\lambda(\theta-1)}) = \int_0^\infty e^{-\lambda(1-\theta)} \frac{p^a}{\Gamma(a)} \lambda^{a-1} e^{-p\lambda} d\lambda = \frac{p^a}{(p+1-\theta)^a}$

$$\begin{aligned} E_\theta(E_\lambda(e^{\lambda(\theta-1)})) &= E_\theta(\frac{p^a}{(p+1-\theta)^a}) \\ &= \int_0^1 \frac{p^a}{(p+1-\theta)^a} \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)} d\theta \\ &= \frac{p^a}{(p+1)^a B(\alpha,\beta)} \int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}(1-\frac{\theta}{p+1})^{-a} d\theta \\ &= \frac{p^a}{(p+1)^a} {}_2F_1(a,\alpha;\alpha+\beta;\frac{1}{p+1}) \end{aligned}$$

- $\Rightarrow$ accept the batch when $\frac{p^a}{(p+1)^a} {}_2F_1(a,\alpha;\alpha+\beta;\frac{1}{p+1}) > \frac{1}{c+1}$

BATCH ACCEPTANCE MODEL

Scenario B: Each fault counts

- Batch with m items in a period
- Loss depending on the number of included faulty items
- Loss function given by

		Batch of m Items		
		All Acceptable	x Acceptable	
		$p = \theta^m$	$p = \binom{m}{x} \theta^x (1-\theta)^{m-x}$	**Exp. Loss**
D's Decision	Accept, d_0	0	$(m-x)\, c'$	$m\, c' (1-\theta)$
	Reject, d_1	c	0	$c\, \theta^m$

BATCH ACCEPTANCE MODEL

- Suppose batch size m known to Defender $D \Rightarrow \lambda$ not relevant
- Expected losses of both decisions

$$l_D(d_0) = E_\theta\left[m\, c'\,(1-\theta)\right] = m\, c'\,(1 - E_\theta\left[\theta\right])$$
$$l_D(d_1) = E_\theta\left[c\,\theta^m\right] = c\, E_\theta\left[\theta^m\right]$$

- Decision: accept the batch (d_0) if and only if

$$m\, c'\,(1 - E_\theta\left[\theta\right]) \leq c\, E_\theta\left[\theta^m\right] \quad \Longleftrightarrow \quad \frac{E_\theta\left[\theta^m\right]}{m} \geq \frac{c'}{c}\,(1 - E_\theta\left[\theta\right])$$

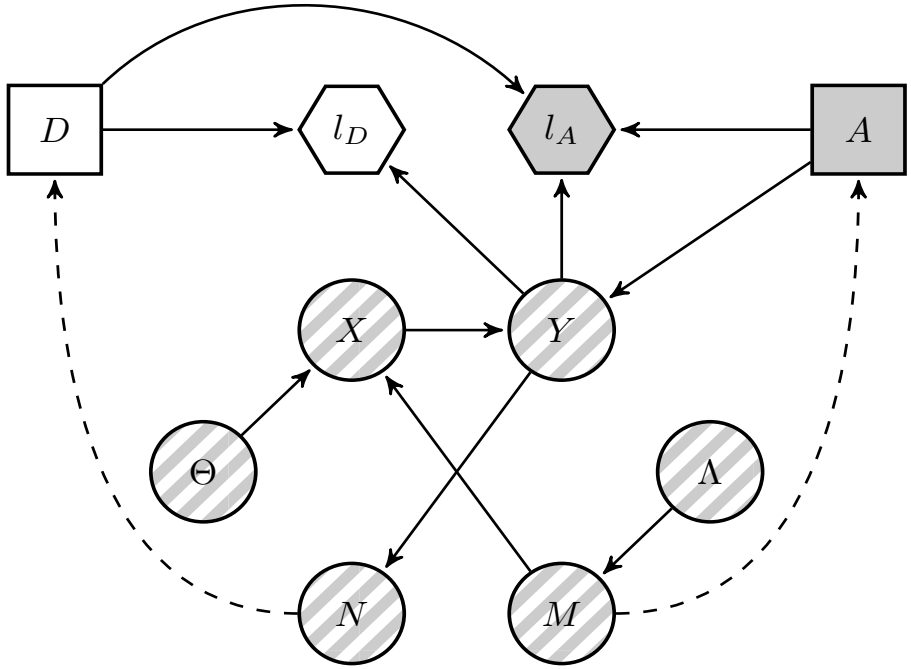
- $E_\theta\left[\theta^m\right]$ decreases as m increases $\Rightarrow$ threshold value $m_B \Rightarrow$ rejection of the batch (d_1) if $m > m_B$
- m_B recursively obtained for posterior $\mathcal{B}e(\alpha + s, \beta + r - s)$ on θ as the smallest integer satisfying

$$\frac{E_\theta\left[\theta^m\right]}{m} \leq \frac{c'}{c}\,\frac{\beta + r - s}{\alpha + \beta + r}$$

ADVERSARIAL BATCH ACCEPTANCE MODEL

- Attacker might alter the batch X to Y and, thus, perturb the data flow process to confound the Defender and reach some objectives
- Batch of size m, with m known by Attacker A
- Attacker A might add items to get a final batch of size n
- Defender D observes n before making her decision
- Gain bigger for A if D accepts one of A's faulty items rather than a faulty item from another source

ADVERSARIAL BATCH ACCEPTANCE MODEL



ADVERSARIAL BATCH ACCEPTANCE MODEL

We study three possible attack strategies, identifying

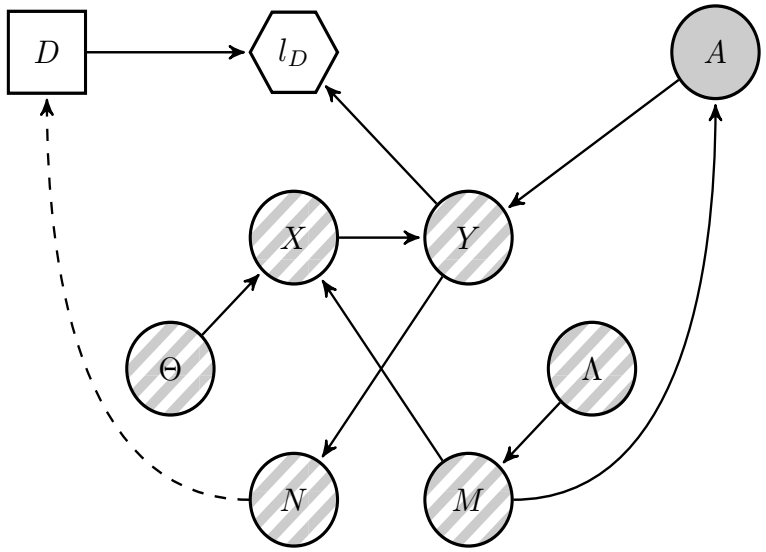
- Attacker's decision variables
- how the item arrival process changes
- Attacker's loss function
- how to solve the problem

The strategies are:

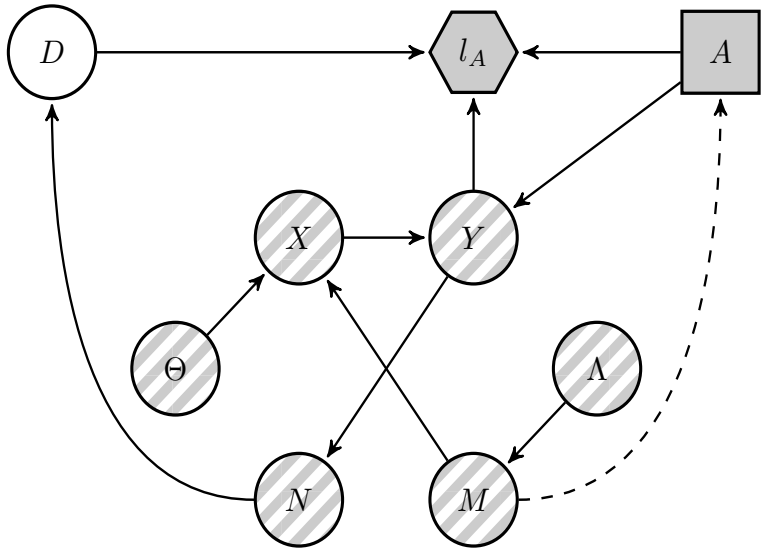
- $\mathcal{S}_1$. Attacker adds some, new faulty items
- $\mathcal{S}_2$. Attacker modifies few original items converting them into faulty ones
- $\mathcal{S}_3$. Attacker combines strategies $\mathcal{S}_1$ and $\mathcal{S}_2$

ADVERSARIAL BATCH ACCEPTANCE MODEL

- n: number of items in a batch observed by Defender D
- x: acceptable items in the batch
- $m - x$: original faulty items (O-faults)
- $n - m$: faulty items produced by the Attacker A (A-faults)



(a) Defender's problem



(b) Attacker's problem

ADVERSARIAL BATCH ACCEPTANCE MODEL

$\mathcal{S}_1$. Attacker adds y_1 new faulty items

- $m + y_1$ data received by Defender include
  - x acceptable items
  - $m - x$ O-faults
  - y_1 A-faults
- Attacker needs to decide y_1, which is random to Defender
- Suppose first that Defender knows $p_D(y_1|m)$, distribution of $Y_1|m$
- Loss structure for Defender

		Final Batch of n Items		
		All Acceptable	Some Faulty	
		$p = q_1(n\|\lambda)$	$p = 1 - q_1(n\|\lambda)$	**Exp. Loss**
D's Decision	Accept, d_0	0	1	$1 - q_1(n\|\lambda)$
	Reject, d_1	c	0	$c\, q_1(n\|\lambda)$

ADVERSARIAL BATCH ACCEPTANCE MODEL

- $n = m + y_1$
- Probability of having a final batch of n items reflects all possible initial sizes of the batch and included faulty items, not just m and y_1, respectively:

$$p_1(n|\lambda) = \sum_{m=0}^{n} p_D(m|\lambda)\, p_D(y_1 = n - m|m)$$

- Probability that all items are acceptable (i.e., $x = m = n$ and $y_1 = 0$)

$$q_1(n|\lambda) = \frac{p_D(m = n|\lambda)\, p_D(y_1 = 0|m = n)}{p_1(n|\lambda)}\, \theta^n$$

- λ relevant here since it provides information on m

ADVERSARIAL BATCH ACCEPTANCE MODE

		Final Batch of n Items		
		All Acceptable	Some Faulty	
		$p = q_1(n\|\lambda)$	$p = 1 - q_1(n\|\lambda)$	**Exp. Loss**
D's Decision	Accept, d_0	0	1	$1 - q_1(n\|\lambda)$
	Reject, d_1	c	0	$c\, q_1(n\|\lambda)$

- Expected losses of both decisions

$$l_D(d_0) = 1 - E_\theta \left[E_\lambda \left[q_1(n|\lambda) \right] \right]$$

$$l_D(d_1) = c\, E_\theta \left[E_\lambda \left[q_1(n|\lambda) \right] \right]$$

- Decision: accept the batch (d_0) if and only if

$$E_\theta \left[E_\lambda \left[q_1(n|\lambda) \right] \right] \geq \frac{1}{1+c}$$

- Decision obtained through simulation

ADVERSARIAL BATCH ACCEPTANCE MODE

- $p_D(y_1|m)$ (and thus $q_1(n|\lambda)$) unknown to Defender $D \Rightarrow$ use ARA
- $x \in \{0, 1, \ldots, m\}$ acceptable items
- $y_1 \in \{0, 1, \ldots\}$ added A-faults
- h unitary gain (for A) due to each O-fault
- g unitary gain (for A) due to each A-fault
- f unitary cost (for A) for adding each A-fault
- Attacker A's loss function, depending on batch composition and decision by D

		Final Batch Composition		
		Acceptable	O-Fault	A-Fault
		x	$m - x$	y_1
D's Decision	Accept, d_0	0	$-h$	$f - g$
	Reject, d_1	0	0	f

ADVERSARIAL BATCH ACCEPTANCE MODE

		Final Batch Composition		
		Acceptable	O-Fault	A-Fault
		x	$m - x$	y_1
D's Decision	Accept, d_0	0	$-h$	$f - g$
	Reject, d_1	0	0	f

- Attacker A's losses associated to Defender D's decisions when A chooses y_1

$$l_A(d_0, y_1, x) = -h\,(m - x) + (f - g)\,y_1$$

$$l_A(d_1, y_1) = f\,y_1$$

ADVERSARIAL BATCH ACCEPTANCE MODE

- Losses: $l_A(d_0, y_1, x) = -h\,(m - x) + (f - g)\,y_1$ and $l_A(d_1, y_1) = f\,y_1$
- Problem faced by A: choose y_1 to minimise expected loss for original batch size m

$$\begin{aligned}
\psi_A(y_1|m) &= p_A(d_0|m + y_1) \int \left(\sum_{x=0}^{m} p_A(x|m, \theta)\, l_A(d_0, y_1, x) \right) p_A(\theta)\, \mathrm{d}\theta \\
&\quad + (1 - p_A(d_0|m + y_1))\, l_A(d_1, y_1) \\
&= y_1\,(f - g\,p_A(d_0|m + y_1)) \\
&\quad - h\,p_A(d_0|m + y_1) \int \left(\sum_{x=0}^{m} p_A(x|m, \theta)\,(m - x) \right) p_A(\theta)\, \mathrm{d}\theta,
\end{aligned}$$

- $p_A(d_0|m + y_1)$ reflects A's beliefs about D's decision d_0 to accept the batch given that she knows the batch size is $n = m + y_1$

ADVERSARIAL BATCH ACCEPTANCE MODE

- Defender does not know Attacker's probabilities and parameters of his loss function
  $\Rightarrow (F, G, H, P_A(d_0|n), P_A(\theta), P_A(x|m,\theta))$ random quantities

- Look for random optimal attack $Y_1^*(m)$ defined through

$$\arg\min_{y_1} \left\{ \begin{array}{l} y_1 \, (F - G \, P_A(d_0|m + y_1)) \\ \\ - H \, P_A(d_0|m + y_1) \int \left(\sum_{x=0}^{m} P_A(x|m,\theta) \, (m - x) \right) P_A(\theta) \, \mathrm{d}\theta \end{array} \right.$$

- Draw from random quantities and get sample $\{Y_{1k}^*(m)\}_{k=1}^K$ of size K from $Y_1^*(m)$

- Estimate $\widehat{p}_D(y_1|m) = P(y_1^*(m) = y_1) \approx \#\{Y_{1k}^*(m) = y_1\}/K$
  $\Rightarrow$ get the optimal amount of added faulty items (e.g. from the mode)

ADVERSARIAL BATCH ACCEPTANCE MODE

Typical assumptions about Attacker's random utilities and probabilities

- Gains and costs uniformly distributed:
  - $F \sim \mathcal{U}(f_1, f_2)$
  - $G \sim \mathcal{U}(g_1, g_2)$
  - $H \sim \mathcal{U}(h_1, h_2)$
- $P_A(x|m, \theta)$ Binomial distribution $\mathcal{B}in(m, \theta)$ (i.e. not a random distribution)
- $P_A(\theta)$ from a Dirichlet process with Beta distribution $\mathcal{B}e(\alpha + s, \beta + r - s)$ as base parameter and concentration parameter ρ
- $P_A(d_0|n)$ modelled through a uniform distribution, although this might require further recursion if deeper strategic thinking is considered

ADVERSARIAL BATCH ACCEPTANCE MODE

- Other two strategies:
  - $\mathcal{S}_2$. Attacker modifies few original items converting them into faulty ones
  - $\mathcal{S}_3$. Attacker modifies few original items converting them into faulty ones and adds some new ones
- Very similar approach: not presented here except for the Attacker A's loss function, depending on batch composition and decision by D

ADVERSARIAL BATCH ACCEPTANCE MODE

$\mathcal{S}_2$. Attacker modifies few original items converting them into faulty ones

- h unitary gain (for A) due to each O-fault
- g unitary gain (for A) due to each A-fault
- e unitary cost (for A) for changing any item to make it faulty

		Final Batch Composition		
		Acceptable	O-Fault	A-Fault
		$x - y_2^0$	$m - x - y_2^1$	y_2
D's Decision	Accept, d_0	0	$-h$	$e - g$
	Reject, d_1	0	0	e

ADVERSARIAL BATCH ACCEPTANCE MODE

$\mathcal{S}_3$. Attacker modifies few original items converting them into faulty ones and adds some new ones

- h unitary gain (for A) due to each O-fault
- g unitary gain (for A) due to each A-fault
- e unitary cost (for A) for changing any item to make it faulty
- f unitary cost (for A) for adding each A-fault

		Final Batch Composition			
		Acceptable	O-Fault	A-Fault	
				Injected	*Modified*
		$x - y_2^0$	$m - x - y_2^1$	y_1	y_2
D's Decision	Accept, d_0	0	$-h$	$f - g$	$e - g$
	Reject, d_1	0	0	f	e

DISCUSSION

- New ARA approach to dealing with the AHT problem
- Symmetric losses and strong common knowledge assumptions typical of non-cooperative game theory have been avoided
- Multiple Attackers and/or multiple Defenders cases in the AHT problem are also of interest
  - need to differentiate when Attackers are completely independent or totally coordinated or are such that their attacks influence somehow each other
  - possibility of several Defenders, possibly cooperating but with different observations of the data flow
- New strategies, e.g. Attacker could add (apparently) acceptable items to confound the Defender
- Possible application in adversarial signal processing, such as in Electronic Warfare where pulse/signal environment is generally very complex with many different radars transmitting simultaneously and signals possibly jammed by hostile radars

ACCEPTANCE SAMPLING

Work stemming from Lindley and Singpurwalla (1991)

- Manufacturer M (she) is trying to sell a batch of items to a consumer C (he) who may either accept ($\mathcal{A}$) or reject ($\mathcal{R}$) the batch provided by M
- C's decision depends on the evidence provided by M to C, based on a sample from an inspection that M may perform
- The decision M faces is whether to offer a sample to C and, if so, the size of such sample
- Both M and C are assumed to be expected utility maximisers
- Lindley and Singpurwalla assume that M, who decides before C, knows C's preferences and beliefs, as well as they share other relevant distributions, a too strong common knowledge assumption
- ARA allows us to overcome such issue (for Bernoulli acceptance sampling problem)
- Addressed also a life testing problem

ACCEPTANCE SAMPLING: GAME THEORY

Sequential problem

- M decides the sample size n to offer to C ($\Rightarrow C$ knows n)
- C has available
  - $p_C(\theta)$, i.e., beliefs about the product quality θ
  - $p_C(d|\theta, n)$, i.e., beliefs about the experiment result d (number of defective items) given θ and decision n of M
  - $u_C(c, \theta)$, i.e., utility function based on decision c: accept ($\mathcal{A}$) or reject ($\mathcal{R}$) the batch

ACCEPTANCE SAMPLING: GAME THEORY

- C computes for each d and n
  - Posterior distribution $p_C(\theta|d,n) \propto p_C(\theta)p_C(d|\theta,n)$
  - Expected utility $\psi_C(d,n,c) = \int u_C(c,\theta)p_C(\theta|d,n)d\theta$
  - Optimal decision c, given d and n:

$$c^*(d,n) = \underset{c\in\{\mathcal{A},\mathcal{R}\}}{\arg\max}\ \psi_C(d,n,c)$$

- All the above known by M who switches to her problem

ACCEPTANCE SAMPLING: GAME THEORY

M knows $p_C(\theta|d,n)$, $\psi_C(d,n,c)$ and $c^*(d,n)$ for each d and n

- M has available
  - $p_M(\theta)$, i.e., beliefs about the product quality θ
  - $p_M(d|\theta,n)$, i.e., beliefs about the experiment result d (number of defective items) given θ and decision n of M
  - $u_M(c,\theta)$, i.e., utility function based on decision c: accept ($\mathcal{A}$) or reject ($\mathcal{R}$) the batch

ACCEPTANCE SAMPLING: GAME THEORY

- M computes for each d and n
  - $\psi_M(n, d, \theta) = u_M(c^*(d, n), n, \theta)$, i.e., utility based on C's decision (known under the common knowledge assumption)
  - $\psi_M(n, \theta) = \int \psi_M(n, d, \theta) p_M(d|\theta, n)\ dd$, i.e., expected utility (w.r.t. d)
  - $\psi_M(n) = \int \psi_M(n, \theta) p_M(\theta)\, d\theta$, i.e., expected utility (w.r.t. θ)
  - $n^* = \arg\max \psi_M(n)$, i.e. optimal decision by M

ACCEPTANCE SAMPLING: ARA

- $p_M(\theta)$, $p_M(d|\theta, n)$ and $u_M(c, n, \theta)$ available as before
- Earlier $c^*(d, n)$ was known but now $p_M(c|d, n)$ is needed (and its computation requires thinking about C's behaviour)
- $\Rightarrow$ Need to compute $\psi_M(n, d, \theta) = \sum_{c \in \{\mathcal{A}, \mathcal{R}\}} u_M(c, n, \theta) p_M(c|d, n)$ to get rid of c
- $p_C(\theta)$, $p_C(d|\theta, n)$, and $u_C(c, \theta)$ unknown to M (no common knowledge)
- $\Rightarrow$ random utilities and probabilities generated from $F = (U_C(c, \theta), P_C(\theta), P_C(d|\theta, n))$
- Computation of random functional $\Psi^*_C(d, n, c) = \int U_C(c, \theta) P_C(\theta) P_C(d|\theta, n) d\theta$
- Computation of the random optimal alternative, given d and n:

$$C^*(d, n) = \underset{c \in \{\mathcal{A}, \mathcal{R}\}}{\arg\max} \ \Psi^*_C(d, n, c)$$

- $\Rightarrow$ empirical distribution of $C^*(d, n)$ to estimate $p_M(c|d, n)$

BERNOULLI ACCEPTANCE SAMPLING

The manufacturer's viewpoint

- Sample of size n offered by manufacturer, possibly defective with probability θ
- Sampling model binomial for d defective items with $p_M(d|\theta, n) \sim Bin(n, \theta)$
- θ with a beta distribution $p_M(\theta) \sim \beta e(\beta_1, \beta_2)$
- Utility function $u_M(c, n, \theta)$ as in Lindley and Singpurwalla (1991):
  - $u_M(\mathcal{A}, n, \theta) = b_1 + b_2\theta + b_4 n$,
  - $u_M(\mathcal{R}, n, \theta) = b_3 + b_4 n$
  - b_4 unit cost of providing each sample unit
  - b_2 penalty for defectiveness; the higher θ, the worse the corresponding cost
  - $b_1 > b_3$: preference for accepted items rather than rejected
  - $b_3 > b_1 + b_2$: preference for rejection rather than acceptance of very low quality lot (for reputation)

BERNOULLI ACCEPTANCE SAMPLING

Assumptions on C

- Same sampling model binomial for d defective items with $p_M(d|\theta, n) \sim Bin(n, \theta)$
- Random distribution $P_C(\theta)$ given by
  - Beta distribution $p_c(\theta) \sim \beta e(\alpha_1, \alpha_2)$
  - Uniform distributions $\alpha_1 \sim \mathcal{U} \in [a_{11}, a_{12}]$, and $\alpha_2 \sim \mathcal{U} \in [a_{21}, a_{22}]$
  - Compare with Lindley and Singpurwalla (1991) who considered $p_c(\theta) \sim \beta e(\alpha_1, \alpha_2)$, with known α_1 and α_2
- Random utility $U_C(c, \theta)$, similar to Lindley and Singpurwalla (1991):
  - $u_C(\mathcal{A}, \theta) = a_1 + a_2\theta$,
  - $u_C(\mathcal{R}, \theta) = a_3$,
  - where $a_1 > a_3 > a_1 + a_2$ and $a_2 < 0$

BERNOULLI ACCEPTANCE SAMPLING

An example (values of the parameters omitted)

		$n=0$	1	2	3	4	5	6	7	...
$\widehat{p_M}(\mathcal{A}\|d,n)$	$d=0$	x	0.4	0.49	0.55	0.61	0.65	0.68	0.71	
	$d=1$	x	0.22	0.34	0.42	0.49	0.54	0.58	0.62	
	$d=2$	x	x	0.19	0.29	0.37	0.44	0.49	0.53	
	$d=3$	x	x	x	0.16	0.26	0.33	0.4	0.45	
	...	x	x	x	x	0.14	0.23	0.3	0.36	

Acceptance probabilities for various manufacturer decisions and experimental results

BERNOULLI ACCEPTANCE SAMPLING

	$n = 1$	2	3	4	5	6
$\psi_M(n)$	4.25	4.325	4.374	4.408	4.43	4.444
...	7	8	9	10	11	12
$\psi_M(n)$	4.453	4.456	4.457	4.456	4.451	4.444

Expected utilities of various manufacturer decisions ($n = 9$ optimal decision)

CLASSIFICATION

- Classification consists in assigning instances from a given domain X, described by a set of discrete- or continuous-valued attributes, into a set of classes C
- A good performance measure for classifiers is the misclassification error which is the fraction of instances that are misclassified by the model
- Not all the attributes might be available for all instances (e.g. temperature could be missing): the incomplete dataset could be either ”repaired” in a preprocessing phase, or handled in some special way by modeling algorithms
- Data might be corrupted by some noise: classifiers are not always able to detect and then handle the noise
- To take the simplest example, do two instances with exactly the same attribute values but different class labels result from noise or rather from an insufficient set of attributes which cannot fully differentiate instances from different classes?
- Such questions can be often asked, but rarely answered, unless we accept a somewhat evasive answer that both hypotheses represent simply two different views on the same phenomenon

BAYESIAN CLASSIFIERS

- Naive Bayes is a simple probabilistic classifier based on applying Bayes Theorem with strong independence (naive) assumptions
- The basic idea of Bayes Theorem is that the outcome of a hypothesis or an event can be predicted based on some evidences that can be observed
- In classification, the goal is to classify an instance based on its features
- Typically, the more evidences we can gather, the better the classification accuracy can be obtained
- Cichosz, P. (2015), *Data Mining Algorithms: Explained Using R*, Wiley

```
install.packages("remotes")
remotes::install_github("42n4/dmr.util",force=TRUE)
remotes::install_github("42n4/dmr.data",force=TRUE)
library(dmr.util)
library(dmr.data)
library(dmr.stats)
```

BAYESIAN CLASSIFIERS

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B)+\mathbb{P}(A|B^C)\mathbb{P}(B^C)} \\ &= \frac{.95\cdot.001}{.95\cdot.001+.005\cdot.999} = 0.1598\end{aligned}$$

```
bayes.rule <- function(prior, inv){prior*inv/sum(prior*inv)}
# let P(burglery)=0.001,
# P(alarm|burglery)=0.95,
# P(alarm|not burglery)=0.005
# calculate P(burglery|alarm)
bayes.rule(c(0.001, 0.999), c(0.95, 0.005))
```

- Bayes rule for an exhaustive set of mutually exclusive events
- Partition $\{A_1,\ldots,A_n\}$ of Ω and $B\subset\Omega : \mathbb{P}(B)>0$

$$\mathbb{P}(A_i|B)=\frac{\mathbb{P}(B|A_i)P(A_i)}{\sum_{j=1}^{n}\mathbb{P}(B|A_j)P(A_j)}$$

CLASSIFICATION

	outlook	temperature	humidity	windy	play
1	sunny	hot	high	false	N
2	sunny	hot	high	true	N
3	overcast	hot	high	false	P
4	rain	mild	high	false	P
5	rain	cool	normal	false	P
6	rain	cool	normal	true	N
7	overcast	cool	normal	true	P
8	sunny	mild	high	false	N
9	sunny	cool	normal	false	P
10	rain	mild	normal	false	P
11	sunny	mild	normal	true	P
12	overcast	mild	high	true	P
13	overcast	hot	normal	false	P
14	rain	mild	high	true	N

- Training set, in `weather.txt`, where the first four attributes describe weather conditions and the last one classifies them as appropriate or not for playing sports

- Dataset unrealistic and unsuitable for evaluating the performance of classification algorithms because of very small sample size, but good for illustrating the concepts

BAYESIAN CLASSIFIERS

- The class-probability approach to Bayesian inference applies the Bayes rule to calculate probabilities of the following form:

$$P(c = d|a_1 = v_1, a_2 = v_2, \ldots, a_n = v_n),$$

  i.e. the probability of an instance belonging to class d if its attribute values are $v_1, v_2, \ldots, v_n$, respectively

- The goal is to find the class d which maximizes

$$P(c = d|a_1 = v_1, \ldots, a_n = v_n) = \frac{P(c = d)P(a_1 = v_1, \ldots, a_n = v_n|c = d)}{P(a_1 = v_1, \ldots, a_n = v_n)}$$

- Since the denominator is the same for all classes since it does not depend on a class, the optimization is performed only for the numerator

- We could choose $P(c = d) = \frac{|T^d|}{|T|}$, where $|T|$ is the number of instances in the training phase and $|T^d|$ are those belonging to the class d

```
weather=read.table("weather.txt",header=T)
pdisc(weather$play) # prior class probabilities
```

BAYESIAN CLASSIFIERS

- The conditional joint probability of attribute values given the class cannot be directly estimated from even a perfectly representative training set of any realistic size

- For most practical datasets, there are numerous attribute value combinations not appearing at all, and most of those that do appear, appear exactly once

- This would lead as to estimating probabilities for most attribute value combinations as 0 or $1/|T|$

- Such estimates would be clearly useless for classification, because they do not allow one to differentiate probabilities for instances with different attribute values

- Therefore, the conditional joint probability of attribute values given the class is calculated as the product of per-attribute marginal conditional probabilities:

$$P(a_1 = v_1, \ldots, a_n = v_n | c = d) = \prod_{i=1}^{n} P(a_i = v_i | c = d)$$

BAYESIAN CLASSIFIERS

- The assumption holds only if the attributes are conditionally independent given the class, which is unfortunately usually not true

- The naïve Bayes classifier adopts this independence assumption, ignoring the fact that it is rarely satisfied, and this gives the name ”naïve”

- Probabilities of attribute values given the class estimated from training data

$$P(a_i = v_i | c = d) = \frac{|T^d_{a_i=v_i}|}{|T^d|}$$

where $|T^d_{a_i=v_i}|$ is the number of instances in d for which the value of attribute a_i is v_i and $|T^d|$ are those belonging to the class d

```
# conditional attribute value probabilities given the class
pcond(weather$outlook, weather$play)
pcond(weather$temperature, weather$play)
pcond(weather$humidity, weather$play)
pcond(weather$wind, weather$play)
```

BAYESIAN CLASSIFIERS

```
> pdisc(weather$play)
        N         P
0.3571429 0.6428571
> pcond(weather$outlook, weather$play)
           N         P
overcast 0.0 0.4444444
rain     0.4 0.3333333
sunny    0.6 0.2222222
>    pcond(weather$temperature, weather$play)
       N         P
cool 0.2 0.3333333
hot  0.4 0.2222222
mild 0.4 0.4444444
>    pcond(weather$humidity, weather$play)
         N         P
high   0.8 0.3333333
normal 0.2 0.6666667
>    pcond(weather$wind, weather$play)
        N         P
false 0.4 0.6666667
true  0.6 0.3333333
```

BAYESIAN CLASSIFIERS

- To achieve the desired capability of calculating the conditional class probability given attribute values it is sufficient to estimate the following probabilities from the training set:
  - $P(c = d)$ for each class $d \in C$
  - $P(a_i = v_i | c = d)$ for each class $d \in C$, each attributea_i and each value $v_i \in A_i$
- This set of probabilities constitutes model representation for the naïve Bayes classifier
- To create such a model, one needs to estimate all these probabilities from the training set, which reduces to simple counting required to obtain $|T^d_{a_i=v_i}|$ and $|T^d|$
- These counts can be calculated by a single iteration through the training set

BAYESIAN CLASSIFIERS

```
data(weather,package="dmr.data")# Better to get weather this way
## create a naive Bayes classifier
nbc <- function(formula, data){class <- y.var(formula)
attributes <- x.vars(formula, data)
cc <- integer(nlevels(data[[class]]))  # initialize class counts
names(cc) <- levels(data[[class]])
avc <- sapply(attributes, # initialize attribute-value-class counts
function(a)
matrix(0, nrow=nlevels(data[[a]]), ncol=nlevels(data[[class]]),
         dimnames=list(levels(data[[a]]), levels(data[[class]]))))
for (i in (1:nrow(data))) # iterate through training instances
{cc[data[[class]][i]] <- cc[data[[class]][i]]+1 #increment class count
  for (a in attributes)   # increment attribute-value-class counts
    avc[[a]][data[[a]][i],data[[class]][i]] <-
      avc[[a]][data[[a]][i],data[[class]][i]]+1}
# calculate probability estimates based on counts
`class<-`(list(prior=cc/sum(cc), cond=sapply(avc, function(avc1)
t(apply(avc1, 1, "/", colSums(avc1))))),"nbc")}
nbw <- nbc(play~., weather) # naive Bayes classifier for weather data
nbw
```

BAYESIAN CLASSIFIERS

```
> $prior
       no       yes
0.3571429 0.6428571
> $cond$outlook
          no       yes
overcast 0.0 0.4444444
rainy    0.4 0.3333333
sunny    0.6 0.2222222
> $cond$temperature
      no       yes
cold 0.2 0.3333333
hot  0.4 0.2222222
mild 0.4 0.4444444
> $cond$humidity
        no       yes
high   0.8 0.3333333
normal 0.2 0.6666667
>$cond$wind
        no       yes
high   0.6 0.3333333
normal 0.4 0.6666667
```

BAYESIAN CLASSIFIERS

- Applying the naïve Bayes classifier to predict class probabilities for a given instance x is even more straightforward

- We just need to multiply the prior class probability and the conditional probabilities of the instance's attribute values given the class, i.e. use $P(a_i = v_i|c = d)$ with $v_i = a_i(x)$:

$$P(d|x) = \frac{1}{b} P(c = d) \prod_{i=1}^{n} P(a_i = a_i(x)|c = d)$$

- The normalizing constant b is given by

$$b = \sum_{d' \in C} P(c = d') \prod_{i=1}^{n} P(a_i = a_i(x)|c = d')$$

- Note that all probabilities needed to classify an arbitrary instance are estimated during model construction, and prediction requires just selecting and multiplying an appropriate subset of them, corresponding to the attribute values of the classified instance x

BAYESIAN CLASSIFIERS

```
  ## naive Bayes prediction for a single instance
  predict1.nbc <- function(model, x)
 {
  aind <- names(x) %in% names(model$cond)
  bnum <- model$prior*apply(mapply(function(a, v)
  model$cond[[a]][v,], names(model$cond), x[aind]),
  1, prod)
  bnum/sum(bnum)
}
## naive Bayes prediction for a dataset
predict.nbc <- function(model, data)
{
  t(sapply(1:nrow(data), function(i) predict1.nbc(model, data[i,])))
}
  # make predictions for the weather data
predict(nbw, weather)
cbind(predict(nbw, weather),weather$play)
```

CLASSIFICATION

```
            no        yes
 [1,] 0.79541735 0.20458265 1
 [2,] 0.92103601 0.07896399 1
 [3,] 0.00000000 1.00000000 2
 [4,] 0.46351931 0.53648069 2
 [5,] 0.06716418 0.93283582 2
 [6,] 0.17763158 0.82236842 1
 [7,] 0.00000000 1.00000000 2
 [8,] 0.66032609 0.33967391 1
 [9,] 0.13941480 0.86058520 2
[10,] 0.09747292 0.90252708 2
[11,] 0.42163100 0.57836900 2
[12,] 0.00000000 1.00000000 2
[13,] 0.00000000 1.00000000 2
[14,] 0.72160356 0.27839644 1
```

CLASSIFICATION

- Classification: widely used supervised learning method, applied, e.g., in computer vision, genomics, credit scoring and spam detection
- Currently, a major research area in Statistics and Machine Learning (ML)
- Most efforts focused on obtaining more accurate algorithms
- Less attention for a relevant aspect: presence of adversaries manipulating data to deceive the classifier in order to obtain a benefit (e.g. credentials of bank account)
- Example: Fraud detection
  - ML algorithms developed for detection $\Rightarrow$ fraudsters learn how to evade them
  - Detection more likely for huge transactions $\Rightarrow$ smaller ones more frequently
- No common knowledge $\Rightarrow$ Adversarial Risk Analysis (ARA)

WHY ADVERSARIAL CLASSIFICATION?

- In multiple domains such as malware detection, automated driving systems, or fraud detection, classification algorithms are susceptible to being attacked by malicious agents willing to perturb the value of instance covariates to pursue certain goals

- Such problems pertain to the field of adversarial machine learning and have been mainly dealt with, perhaps implicitly, through game-theoretic ideas with strong underlying common knowledge assumptions

- These are not realistic in numerous application domains in relation to security and business competition

- We present an alternative Bayesian decision theoretic framework that accounts for the uncertainty about the attacker's behavior using adversarial risk analysis concepts

CLASSIFICATION UNDER ATTACKS

- Sentiment analysis problem: assess if a movie had positive or negative reviews
- 2400 IMDb reviews (1200 positive, 1200 negative) extracted from Kotzias et al (2015)
- 150 binary features indicating the presence or absence of the most common words in the dataset
- Label indicating positive ($y = 0$) or negative ($y = 1$) review
- Attacker A: manipulate positive reviews $\Rightarrow$ classified as negative ones
- Attacker B: manipulate negative reviews $\Rightarrow$ classified as positive ones
- Attacks based on change of two words at most
- Accuracy of 4 classifiers over clean and attacked test data

CLASSIFICATION UNDER ATTACKS

Classifier	**Clean data**	**Attacked data** Attacker A	**Attacked data** Attacker B
Logistic Regression	0.728 ± 0.005	0.322 ± 0.011	0.418 ± 0.010
Naive Bayes	0.722 ± 0.004	0.333 ± 0.009	0.405 ± 0.009
Neural Network	0.691 ± 0.019	0.338 ± 0.021	0.417 ± 0.015
Random Forest	0.720 ± 0.005	0.327 ± 0.011	0.397 ± 0.013

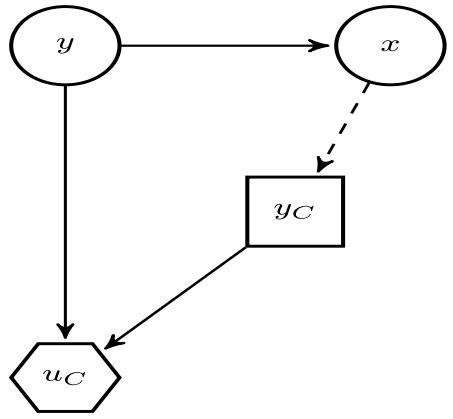
- Accuracy comparison (with precision) of four classifiers on clean and attacked data
- All models trained under same conditions, randomly splitting the dataset into train and test subsets with sizes 90% and 10%, respectively
- Accuracy means and standard deviations are estimated via hold-out validation over 10 repetitions
- MAP (Maximum a posterior) under Gaussian prior for Logistic Regression
- Two hidden layers for Neural Network

ADVERSARIAL HYPOTHESIS TESTING

- Already seen but now it goes in a different direction: classification
- Test of two simple hypotheses: $\Theta = \{\theta_0, \theta_1\}$
- Observation x generated according to a model depending on θ
- x altered to y by A's action a
- y observed by D $\Rightarrow$ D's decision d on θ based on y, without observing x
- Depending on d and actual $\theta \Rightarrow$ losses (utilities) for both agents
- Efforts by A in minimising the loss
- Support for D in choosing θ to minimise the loss

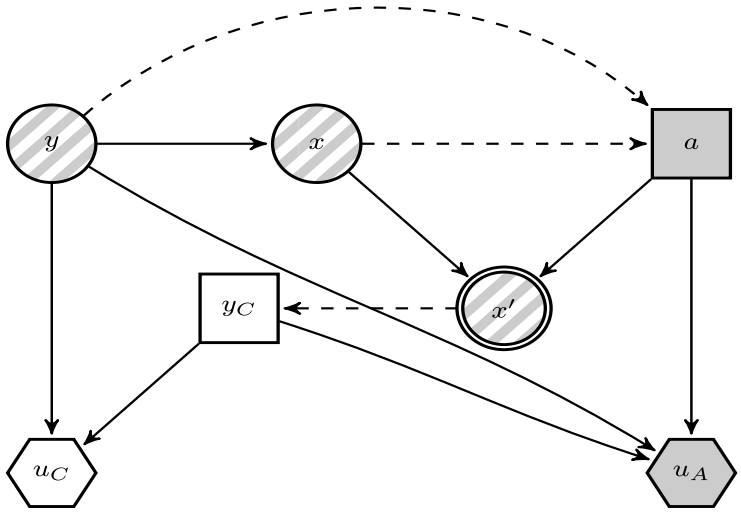
BINARY CLASSIFICATION

- Classifier C receives two types of objects: malicious ($y = +$) or innocent ($y = -$)
- Objects have features x whose distribution depends on their type y
- Classification problems broken down into two separate stages:
  - inference about $p_C(y|x)$, C's beliefs about type given features
  - decision about class assignment y_C, based on $p_C(y|x)$ and utility $u_C(y_C, y)$
- Node: decision (square), uncertainty (circle), deterministic (double), utility (hex.)
- Arrow: conditional relation (solid), information available at decision time (dashed)



ADVERSARIAL CLASSIFICATION

- Adversary A chooses attack a s.t. actual $x \to x' = a(x)$ observed by C
- A attacks only for malicious instances ($y = +$)
- Nodes in bi-agent influence diagram: grey (A), white (C), striped (both A and C)
- Decisions: attack a by A and classification y_C by C
- Utilities: $u_C(y_C, y)$ for C and $u_A(y_C, y, a)$ for A



CLASSIFIER PROBLEM

$$\begin{aligned}
\text{Find class } c(x') &= \arg\max_{y_C} \sum_{y\in\{+,-\}} u_C(y_C, y) p_C(y|x') \\
\text{(ignore } p(x')) &= \arg\max_{y_C} \Big[u_C(y_C, -) p_C(x'|-) p_C(-) \\
&+ u_C(y_C, +) p_C(+) \sum_{x\in\mathcal{X}'} p_C(a_{x\to x'}|x, +) p_C(x|+) \Big]
\end{aligned}$$

- Expected utility maximisation
- $\mathcal{A}(x)$: set of possible attacks for actual x
- $\mathcal{X}' = \{x : a(x) = x' \text{ for some } a \in \mathcal{A}(x)\}$: x's potentially leading to observed x'
- $p_C(y)$: beliefs about the class distribution
- $p_C(x|y)$: beliefs about feature distribution given the class (under no attacks)
- $u_C(y_C, y)$: utility in classifying y_C with actual y
- $p_C(a|x, y)$: beliefs about A's action, given x and y (Think of A's behaviour!)

ATTACKER PROBLEM

- Find optimal attack

$$
\begin{aligned}
a^*(x,y) &= \arg\max_a \int \Big[u_A(+,+,a)\ p + u_A(-,+,a)\ (1-p) \Big] f_A(p|a(x))dp \\
&= \arg\max_a \left[u_A(+,+,a) - u_A(-,+,a) \right] p^A_{a(x)} + u_A(-,+,a)
\end{aligned}
$$

- A: modify x so that C classifies malicious instances as innocent (A's maximum expected utility)
- A: modify only malicious instances, i.e. $y = +$, and not innocent, i.e. $y = -$
- C's decision: uncertain for A
- $u_A(y_C, y, a)$: utility for A when C says y_C, actual label is y and the attack is a
- $p_A(c(x')|x')$: A's beliefs about the classification result when C observes x'
- $p = p_A(c(a(x)) = +|a(x))$: A's beliefs about C classifying as malicious after observing $x' = a(x)$
- Uncertainty on p modelled via density $f_A(p|a(x))$ with expectation $p^A_{a(x)}$.

CLASSIFIER PROBLEM

- Find $a^*(x,y) = \arg\max_a \left[u_A(+,+,a) - u_A(-,+,a)\right] p^A_{a(x)} + u_A(-,+,a)$
- C does not know A's utilities u_A and probabilities $p^A_{a(x)}$
- C's uncertainty modelled through random utility U_A and random expectation $P^A_{a(x)}$
- Solve for the random optimal attack, optimising the random expected utility
  $A^*(x,+) = \arg\max_a \left(\left[U_A(+,+,a) - U_A(-,+,a)\right] P^A_{a(x)} + U_A(-,+,a) \right)$
- $\Rightarrow p_C(a_{x\to x'}|x,+) = Pr(A^*(x,+) = a_{x\to x'})$, assuming a discrete set of attacks
- Approximation through simulation of K samples $\left(U^k_A(y_C,+,a), P^{A,k}_{a(x)}\right)$ from random utilities and probabilities
  $\Rightarrow A^*_k(x,+) = \arg\max_a \left(\left[U^k_A(+,+,a) - U^k_A(-,+,a)\right] P^{A,k}_{a(x)} + U^k_A(-,+,a) \right)$
- Estimation: $\widehat{p_C}(a_{x\to x'} \,|\, x,+) = \#\{A^*_k(x,+) = a_{x\to x'}\}/K$

RANDOM UTILITY

- Random utility $U_A(y_C, +, a)$ includes two components
  - A's gain from C's decision
  - random cost B of implementing an attack
- $Y_{y_C y}$: gain when C decides y_C with y actual label
- $-Y_{++} \sim Ga(\alpha_1, \beta_1)$ with expected gain $\alpha_1/\beta_1 = -d$ for A and variance α_1/β_1^2
- $Y_{-+} \sim Ga(\alpha_2, \beta_2)$ with expected gain $\alpha_2/\beta_2 = e$ for A, and variance α_2/β_2^2
- $Y_{+-} = Y_{--} = \delta_0$, Dirac at 0: no gain for A from innocent instances
- $\Rightarrow A$'s gain $(Y_{y_C y} - B)$
- If A risk prone $\Rightarrow U_A(y_C, y, a) = \exp(\rho\,(Y_{y_C y} - B))$ with random risk proneness coefficient $\rho \sim U[a_1, a_2]$, $a_1 > 0$

RANDOM PROBABILITY

- $P^A_{a(x)}$, A's (random) expected probability that C classifies as malicious for $x' = a(x)$
- C guesses A's beliefs about C's classification when observing $x' \Rightarrow$ delicate
- Hierarchy of decisions: A should know what C does when knowing what A does ...
- Probabilities to be specified at each stage until no more available information $\Rightarrow$ non-informative distribution at that stage
- Heuristic at first stage based on $Pr_C(c(x') = +|x') = r$ (C classifies as malicious observing x'), with some uncertainty around it $\Rightarrow P^A_{a(x)} \sim \beta e(\delta_1, \delta_2)$, with mean $\delta_1/(\delta_1 + \delta_2) = r$ and adequate variance
- In general, given observed x', consider all instances leading to it
  - p_1: proportion of instances originally malicious
  - p_2: proportion of instance originally innocent
  - $\Rightarrow r = p_1/(p_1 + p_2)$

SPAM DETECTION

- m emails as *bag-of-words*: binary features about presence (1) or not (0) of n words
- Label indicates whether the message is spam $(+)$ or not $(-)$
- Email as n-dimensional vector $x = (x_1, x_2, .., x_n)$ of 0's or 1's, with label y
- Only word insertion attacks $\Rightarrow$ 0's replaced by 1's
- Interest in insertion of one word at most
- $I(x)$: set of indices s.t. $x_i = 0$ in $x \Rightarrow \mathcal{A}(x) = \{a_0, a_i; \forall i \in I(x)\}$ set of possible attacks with identity a_0 and a_i transforming i-th 0 into 1
- $J(x')$: set of indices with value 1 in x' received by $C \Rightarrow \mathcal{X}' = \{x', x'_j; \forall j \in J(x')\}$ and x'_j message potentially leading to x', with j-th 1 in x' replaced with 0

SPAM DETECTION

- $u_C(y_C, y)$ standard
- $p_C(y)$ and $p_C(x|y)$ standard if considering only exploratory attacks and using a generative classifier (i.e. based on a generative model $p_C(x, y)$) to estimate them
- Strategic component for $p_C(a_{x \to x'}|x, y)$ and use of ARA to approximate it
- Adversary's random utilities obtained as before
- Beta distribution for $P^A_{a(x)}$ with adequate variance and mean r_a
  - $q_0 = p_C(x'|-)p_C(-)$: original label $-$ left unchanged by A
  - $q_j = p_C(x'_j|+)p_C(+), \forall j \in J(x')$: original label $+$ changed by A
  - $q_{n+1} = p_C(x'|+)p_C(+)$: original label $+$ left unchanged by A
  - $r_a = \dfrac{\sum_{i \in J[a(x)]} q_i + q_{n+1}}{q_0 + \sum_{i \in J[a(x)]} q_i + q_{n+1}}$, i.e. mean set equal to C's own probability of classifying malicious for $x' = a(x)$
  - $P^A_{a(x)}$: A's (random) expected probability of C classifying malicious for $x' = a(x)$

SPAM DETECTION

- Spambase Data Set from UCI Machine Learning repository
  - 4601 emails, out of which 1813 are spam
  - 54 relevant words for each email $\Rightarrow$ 54 dimensional vector x of 0's and 1's
  - data randomly split into training (75%) and test (25%) sets, with 100 repetitions
- Training not affected by attacks $\Rightarrow \widehat{p}_C(y)$ and $\widehat{p}_C(x|y)$ from Naive Bayes classifier
- Simulations (sample size 1000) with 4 utilities for C and different variances for random expected probability $P^A_{a(x)}$ (increasing percentages k of maximum value)
- Comparison between ACRA and Naive Bayes: accuracy, utility, false positive (FPR) and false negative rates
- ACRA more robust w.r.t. attacks, identifying more attacked spam emails, even for larger k, i.e. variance, worsening the performance
- ACRA $\Rightarrow$ lower FPR, i.e. less non-spam are rejected as spam (more important than accepting spam)

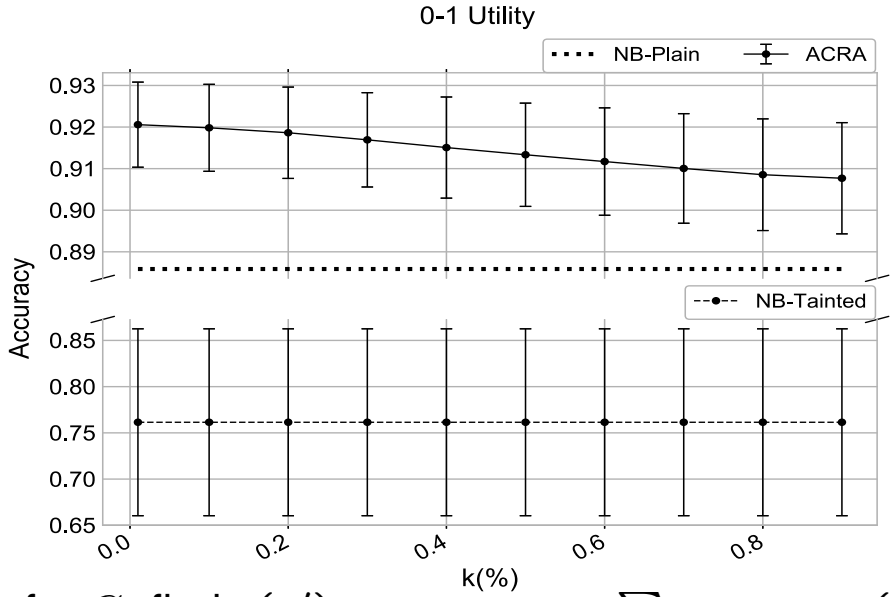
SPAM DETECTION

- Checking utility robustness through 4 utilities for C:
  - $0/1$ Utility $\Rightarrow$ 1 if correctly classified and 0 o.w.
  - Three utilities taking values
    - 1 if correctly classified
    - -1 for spam classified as legit
    - $-2/-5/-10$ for legit classified as spam
- Random utilities for A (m=mean, v=variance)
  - $-U_A(+,+,a) \sim Ga(2500, 0.002) \Rightarrow m = 5, v = 0.01$
  - $U_A(-,+,a) \sim Ga(2500, 0.002) \Rightarrow m = 5, v = 0.01$
  - $U_A(-,-,a) = U_A(+,-,a) = \delta_0$
- Random cost $B = d(a) \cdot \alpha$, with $d(a) = \#$ word changes and $\alpha \sim U[0.4, 0.6]$
- Random risk proneness coefficient $\rho \sim U[0.4, 0.6]$

SPAM DETECTION

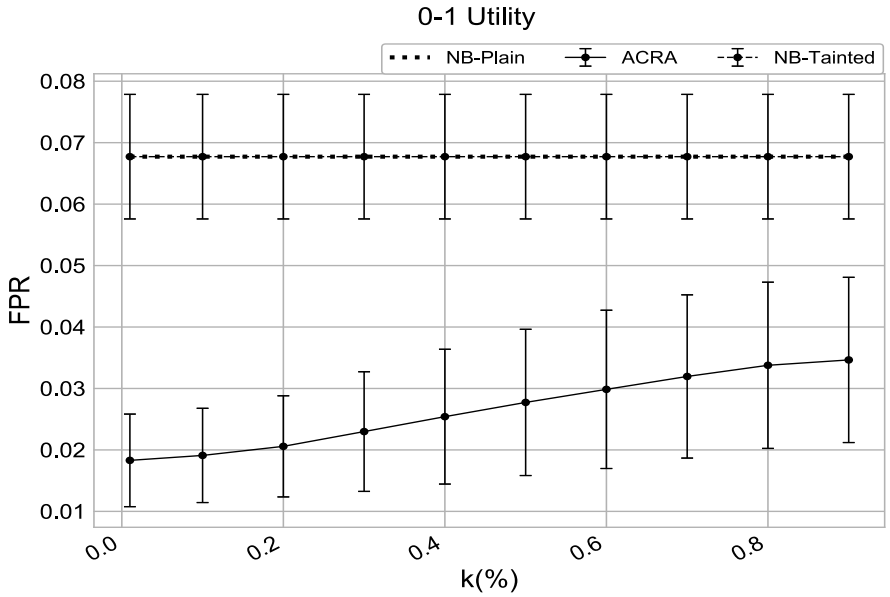
- Beta distribution for $P^A_{a(x)}$ with mean $r = Pr_C(c(a(x)) = +|a(x))$
  - Concave to avoid malicious $a(x)$ concentrated around 0 or 1
  - $\Rightarrow$ variance $\leq \Delta = \min\left\{[r^2(1-r)]/(1+r), [r(1-r)^2]/(2-r)\right\}$
  - Adjustable variance at $k\Delta$ with $k \in \{0.01, 0.1, 0.2, \cdots, 0.9\}$
- $K = 1000$ Monte Carlo sample size

SPAM DETECTION



- Starting problem for C: find $c(x') = \arg\max_{y_C} \sum_{y \in \{+,-\}} u_C(y_C, y) p_C(y|x')$
- $0/1$ utility function, i.e. 1 for correctly classified instance and 0 otherwise
- Naive Bayes: NB-Plain for original data and NB-Tainted for attacked data
- k: percentage of maximum variance for $P^A_{a(x)}$

SPAM DETECTION



- Naive Bayes: NB-Plain and NB-Tainted behave similarly since A is not modifying innocent instances
- Increasing k (and variance for $P^A_{a(x)}$) $\Rightarrow$ increases FPR
- Reducing FPR crucial in spam detection, as filtering out a non-spam is worse than letting spam reach the user

DISCUSSION ABOUT ACRA

- So far ACRA tested with A's distributions centered around the expected values of C's, but it proves quite robust even when moving away
- Changing all words in the spam detection problem $\Rightarrow 2^n$ possible attacks
  - Ad hoc procedure, e.g., changing only one word and from 0 to 1
  - Smaller sample size
  - Approximations, parallelisation
- Further extensions
  - From binary classification to multi-label (e.g. malware: trojan, adware, virus)
  - From exploratory to poisoning attacks, i.e. attacks also during training
  - Attacks not only on malicious instances but also on innocent ones
  - From generative classifiers ($P(X, Y)$) to discriminative ones ($P(Y|X = x)$)

DISCRIMINATIVE CLASSIFIERS

- In the earlier approach (generative classifier) we supposed to know $p(y)$ and $p(x|y)$, e.g. from a classifier applied to the training set
- Here we suppose to know only $p(y|x)$ and address the problem of classifying an instance when x' is observed $\Rightarrow$ solve $\arg\max_{y_C} \psi(y_C)$ where

$$\begin{aligned}\psi(y_C) &= \int_{\mathcal{X}_{x'}} \left(\sum_{y=1}^{k} u(y_C, y) p(y|x = a^{-1}(x') \right) p(x|x')dx \\ &= \sum_{y=1}^{k} u(y_C, y) \left[\int_{\mathcal{X}_{x'}} p(y|x = a^{-1}(x')) p(x|x')dx \right]\end{aligned}$$

- $p(y|x)$ is based on untainted x
- $\mathcal{X}_{x'}$, the set of reasonable instances x leading to x' if attacked
- Optimisation solved via Monte Carlo using sample $\{x_n\}_{n=1}^{N}$ from $p(x|x')$ but ...
- ... there is a problem: we do not know $p(x|x')$ and we have to estimate it

AB-ACRA

- Suppose $p(x)$ unknown and $p(x'|x)$ known as result of strategic thinking, as before, about the possible attacks
- Efficient approach to sample from $p(x|x')$ making use of samples from $p(x'|x)$
- Sample from $p(x|x') \propto p(x'|x)p(x)$ for x and x' discrete
  - Proposal $\tilde{x}$ from transition distribution $q(x \to \tilde{x})$
  - Sampled $\tilde{x}' \sim p(X'|X = \tilde{x})$
  - $\Rightarrow$ accept $\tilde{x}$ if $\tilde{x}' = x'$ with probability $\alpha = \min\left\{1, \frac{p(\tilde{x})q(\tilde{x} \to x_i)}{p(x_i)q(x_i \to \tilde{x})}\right\}$
  - Very slow convergence
- Sample from $p(x|x')$ for x and x' continuous
  - $\tilde{x}$ and $\tilde{x}'$ generated as above
  - Based on Approximate Bayesian Computation (ABC) techniques, accept $\tilde{x}$ if $\phi(\tilde{x}', x') < \epsilon$ for a given distance ϕ and tolerance ϵ
  - For high dimensions, use summary statistics s to accept $\tilde{x}$ if $\phi(s(\tilde{x}'), s(x')) < \epsilon$

ROBUSTIFYING DURING TRAINING

- $p(y|x,\beta)$: model for instances y given covariates x and parameter β
- Training data $\mathcal{D} = \left\{(x_i, y_i)_{i=1}^N\right\}$ and prior $p(\beta)$
  $\Rightarrow$ posterior $p(\beta|\mathcal{D})$ and predictive $p(y|x,\mathcal{D})$
- Using $0-1$ utilities $\Rightarrow \arg\max_{y_C} \iint p(y_C|x,\beta)p(x|x')p(\beta|\mathcal{D})dx\,d\beta$
- Given a sample x, adversarial perturbation $x' \sim p(x'|x)$ and adequate objective function $\mathcal{L}(\beta, x, y)$ (mathematical details and conditions skipped)
  $\Rightarrow$
  - $p(y, x|\beta) \approx p(y, x'|\beta)$, i.e. x and x' affect y similarly
  - $p(y|x,\beta) \approx p(y|x',\beta) \Rightarrow \arg\max_{y_C} \iint p(y_C|x',\beta)p(x'|x)p(\beta|\mathcal{D})dx'd\beta$
  - $p(x|\beta) \approx p(x'|\beta) \Rightarrow \arg\max_{y_C} \iint p(y_C|x',\beta)p(\beta|\mathcal{D})d\beta$

ROBUSTIFYING DURING TRAINING

Classifier	**Attacked (Raw)** Attacker A	**Attacked (CK)** Attacker A	**Attacked (AB-ACRA)** Attacker A
Logistic Regression	0.315 ± 0.007	0.499 ± 0.008	$\mathbf{0.513} \pm 0.008$
Naive Bayes	0.325 ± 0.007	0.645 ± 0.025	$\mathbf{0.665} \pm 0.024$
Neural Network	0.389 ± 0.024	0.592 ± 0.032	$\mathbf{0.638} \pm 0.030$
Random Forest	0.313 ± 0.009	$\mathbf{0.720} \pm 0.013$	0.710 ± 0.017

Classifier	**Attacked (Raw)** Attacker B	**Attacked (CK)** Attacker B	**Attacked (AB-ACRA)** Attacker B
Logistic Regression	0.412 ± 0.004	0.713 ± 0.008	$\mathbf{0.760} \pm 0.011$
Naive Bayes	0.406 ± 0.008	0.783 ± 0.039	$\mathbf{0.800} \pm 0.035$
Neural Network	0.437 ± 0.014	$\mathbf{0.727} \pm 0.050$	0.725 ± 0.052
Random Forest	0.402 ± 0.005	0.779 ± 0.011	$\mathbf{0.782} \pm 0.008$

- Accuracy comparison (with precision) of four classifiers on attacked data with no defense, CK (Common Knowledge) defense and AB-ACRA defense

CONCLUSIONS ABOUT ACRA

- Here more emphasis on modelling and conceptual aspects while the paper contains many details about algorithmic ones, especially about scalability
- Like in ABC, the choice of summary statistics in AB-ACRA might be critical
- Adaptive attackers can be dealt with changing random probability and random utility accordingly
- Here we have considered attacks to i.i.d. sequences but data could come, say, from an autoregressive model

ADVERSARIAL SOFTWARE TESTING

- Software subject to (possibly expensive and dangerous) failures in programming or system design
- $\Rightarrow$ software must undergo rigorous testing, both during development and operation, to verify its reliability
- Optimal policies for software release $\Rightarrow$ important issue in software engineering
- Challenges due to several, often uncertain, complicating factors
- Endogenous factors
  - number of bugs in the software
  - skill in detecting bugs
- Exogenous factors
  - release decisions made by competitors
  - eventual purchasing decision by software buyers

ADVERSARIAL SOFTWARE TESTING

- Monetary aspects
  - costs related to time on test
  - costs related to bugs discovering and their fixing during testing
  - costs related to bugs discovering and their fixing after the release
  - monetary gain for the software sale
- Reputational aspects
- Early software release $\Rightarrow$ larger commercial advantage over competitors
- Less intensely tested software $\Rightarrow$ possible lower quality $\Rightarrow$ potential advantage to competitors

ADVERSARIAL SOFTWARE TESTING

- Singpurwalla and Wilson (2012): Review of software reliability and testing
- Anand, Singh, Das (2015): evaluation of two types (simple and serious) failures in successive versions of a software, during testing and operational phases
- Wilson and O'Riordain (2018): optimal release policy of new versions of Mozilla Firefox based on bug detection data
- Saraf and Iqbal (2019): software reliability model based on NHPP, performing fault detection, observation and correction in two stages and multiple versions
- Mishra, Kapur, Srivastava (2018): reliability growth of software over multiple versions
- Kenett, Ruggeri, Faltin (2018): thorough review of analytic methods in systems and software testing
- Ay, Landon, Ruggeri, Soyer (2022): software testing with possible introduction of bugs

ADVERSARIAL SOFTWARE TESTING

- Ruggeri, Soyer (2018): overview of games and decision models for software testing
- Forman, Singpurwalla (1977, 1979) and Okumoto, Goel (1979): introduction of stopping time models to support software release decisions
- Dalal, Mallows (1988): pioneer work on decision theoretic models for release
- Morali, Soyer (2003): sequential Bayesian decision theoretic setup for developing optimal stopping policies for software testing
- Zeephongsekul, Chiera (1995): first game theoretic approach looking for optimal release policies through Nash equilibrium
  - Dohi, Teraoka, Osaki (2000): different approach since previous solution restricted to particular case and computationally intractable
  - Saito, Dohi (2022): uncovered faults in the earlier two papers showing the existence of Nash equilibrium under some parametric conditions

ADVERSARIAL SOFTWARE TESTING

- Overview of Zeephongsekul and Chiera (1995)
- First work to consider also actions and costs of a competitor
- Two competitors ($i = 1, 2$) produce software performing the same set of tasks and with life cycle length non exceeding T
- Competitor i, $i = 1, 2$, decides to release the software at any time t in $[0, T]$ and sells the product with probability $A_i(t)$ to the only buyer (who buys from one competitor at most)
- $A_i(t)$, $i = 1, 2$, continuously differentiable, concave and s.t. $A_i(0) = A_i(T) = 0$ with a unique maximum at time η_i
  - Choice of $A_i(t)$ not only for mathematical convenience but also justified by actual behaviour
  - Success probability expected to be close to 0 both at the beginning and the end of the life cycle $[0, T]$, because of initial poor reliability and final obsolescence, respectively

ADVERSARIAL SOFTWARE TESTING

- Introduction of expected cost function $c_i(t)$ incurred by player i in releasing the software at time t

- $c_i(t) = c_{1i}t + c_{2i}m(t) + c_{3i}\left(m(T) - m(t)\right)$
  - c_{1i} cost of testing per unit time
  - c_{2i} cost of removing a fault during testing
  - c_{3i} cost of removing a fault during operation, with $c_{3i} > c_{2i}$ since fixing an error is more expensive after release than before it
  - $m(t)$ expected number of faults detected up to time t
  - increasing, concave and differentiable $m(t)$, with $m(0) = 0$

- $\Rightarrow c_i(t)$ convex function with minimum at γ_i s.t. $\Rightarrow m_i^{'}(\gamma_i) = \dfrac{c_{i1}}{(c_{3i} - c_{2i})}$

- T is sufficiently large so that $\gamma_i < T$

ADVERSARIAL SOFTWARE TESTING

- $p_i > 0$: selling price of the software produced by player i
- If player 1 releases software at time x and player 2 at time $y \Rightarrow M_i(x, y)$ is the expected unit profit to player i, with

$$M_1(x,y) = \begin{cases} p_1 A_1(x) - c_1(x) & 0 \leq x < y \leq T \\ p_1(1 - A_2(y))A_1(x) - c_1(x) & 0 \leq y < x \leq T \end{cases}$$

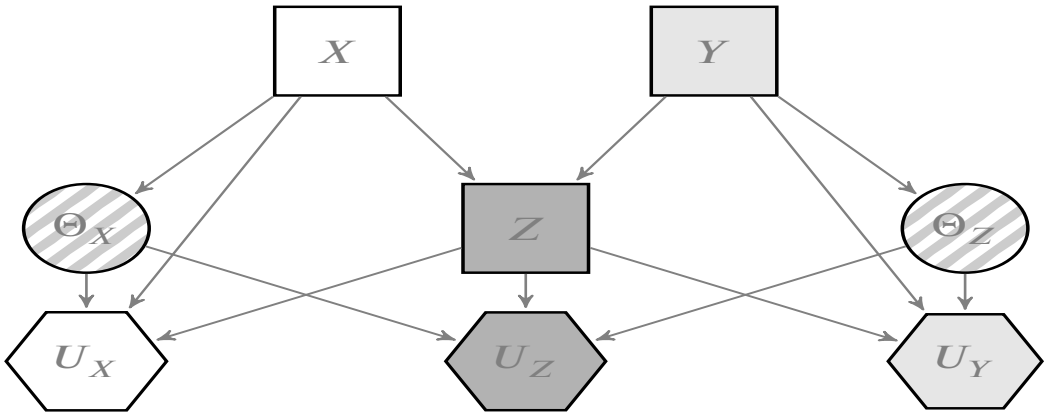
- $M_2(x, y)$ can be described similarly and $M_1(x, y) \neq M_2(x, y)$ in general
- $\Rightarrow$ optimal release policies among Nash equilibrium points in this non-zero sum game (with concerns about the results as mentioned earlier)
- The paper, and all game theoretic work in the field, entails common knowledge assumptions, debatable in competitive business settings as in software development
- $\Rightarrow$ Adversarial Risk Analysis $\Rightarrow$ Adversarial Software Testing

ADVERSARIAL SOFTWARE TESTING

- Support for producer X against competitor Y, trying both to sell software to buyer Z (purchasing from one producer at most)
- X can release the software at any time $x \in [0, T]$
- In absence of competitors, X would succeed in selling the product at the price p_X with probability $A_X(x)$, with $A_X(0) = A_X(T) = 0$ (less restrictive than before)
- Y releases at time $y \in [0, T]$ independently, succeeding to sell at fixed price p_Y with probability $A_Y(y)$, with similar properties as A_X
- Consider a stochastic number $N_X(t)$ of faults found until time t, instead of the expected number $m_X(t) = E[N_X(t)]$
- $N_X(t)$ NHPP with intensity $\lambda_X(t)$ and mean value function $m_X(t) = \int_0^t \lambda_X(u)du$
- Similar definitions apply to Y

ADVERSARIAL SOFTWARE TESTING

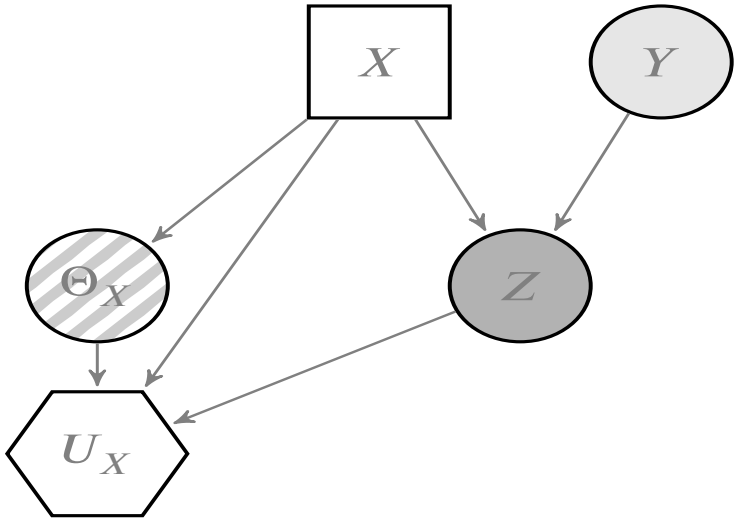
Tri-agent influence diagram representing the basic problem



- Global perspective
- Different colours for different agents
- Square nodes: Decisions by producers (X and Y) and buyer (Z)
- Circle nodes: Uncertain features of X (Θ_X) and Y (Θ_Y), like number of bugs
- Hexagonal nodes: Utilities U_X, U_Y, U_Z for X, Y, Z

ADVERSARIAL SOFTWARE TESTING

Tri-agent influence diagram representing the basic problem



- Perspective from producer X, the one we are taking in the work
- Y's decision now as a circle since it is uncertain for X

ADVERSARIAL SOFTWARE TESTING

- $c_X(t) = c_{1X}t + c_{2X}N_X(t) + c_{3X}\,[N_X(T) - N_X(t)]$
  - c_{1i} cost of testing per unit time
  - c_{2i} cost of removing a fault during testing
  - $c_{3i} > c_{2i}$ cost of removing a fault during operation
- We assume that no new bugs are introduced during the debugging phase
- We assume that fault arrivals can be described by the same process during debugging and operational phase after the software has been released
- There are other assumptions leading to further developments, e.g., price fixed in advance, only two producers, only one buyer, fixed purchase probability

ADVERSARIAL SOFTWARE TESTING

- X and Y release their software at times x and y, respectively ($x \neq y$ a.s.)
- X stops testing if the buyer does not purchase its software, either because it rejects the product or because it has already bought it from Y
- $g_X(x, y)$ (random) gain of producer X given such release times
- Start with $x < y$ and rename g_X as g_{X1}
- $\Rightarrow g_{X1}(x, y) = A_X(x)\,[p_X - c_X(x)] - [1 - A_X(x)]\,[c_{1X}\,x + c_{2X} N_X(x)]$
- First term: expected gain if Z buys X's software given by purchase probability at time x times the difference between selling price and costs due to debugging until x and fault removals after the release up to time T
- Second term: expected loss due to refusal by Z and costs incurred until release time
- Note that $g_{X1}(x, y)$ does not depend on y

ADVERSARIAL SOFTWARE TESTING

- Similarly, Y's gain, for $y < x$, not dependent on x:
- $g_{Y1}(x, y) = A_Y(y)\,[p_Y - c_Y(y)] - [1 - A_Y(y)]\,[c_{1Y}\,y + c_{2Y} N_Y(y)]$
- When $x > y$, the X's gain is renamed as g_{X2}

$$\begin{aligned} g_{X2}(x, y) \quad = \quad & -A_Y(y)\,[c_{1X} y + c_{2X} N_X(y)] + [1 - A_Y(y)]\,\{A_X(x)\,[p_X - c_X(x)] - \\ & - [1 - A_X(x)]\,[c_{1X} x + c_{2X} N_X(x)]\} \end{aligned}$$

- First term: Z buys Y's software and X stops debugging its own
- Second and third term: like earlier, but after Z's refusal of buying Y's software
- Similar result for Y when $y > x$

ADVERSARIAL SOFTWARE TESTING

- Assuming risk neutrality $\Rightarrow$ expected gain $h_X(x,y)$ replacing $N_X(t)$ with its expectation, like for $x < y$

$$\begin{aligned} h_{X1}(x,y) &= A_X(x)\,[p_X - (c_{1X}x + c_{2X}m_X(x) + c_{3X}\,[m_X(T) - m_X(x)])] \\ &\quad - [1 - A_X(x)]\,[c_{1X}x + c_{2X}m_X(x)] \end{aligned}$$

- As an anticipation of what is next, X can also consider $A_Y(y)$ as random and compute its expectation when $x > y$

$$h_{X2}(x,y) = -E(A_Y(y))[c_{1X}y + c_{2X}m_X(y)] + (1 - E(A_Y(y)))\times$$

$$\times[[A_X(x)[p_X - (c_{1X}x + c_{2X}m_X(x) + c_{3X}[m_X(T) - m_X(x)]] - [1 - A_X(x)]\times$$

$$\times[c_{1X}x + c_{2X}m_X(x)]]]$$

- Similar results apply to Y

ADVERSARIAL SOFTWARE TESTING

- $\pi_Y^X(y)$: density modelling X's beliefs about Y's release decision being time y
- Expected gain associated with release decision x
  $M_X(x) = \int h_X(x,y)\pi_Y^X(y)dy = \int_0^x h_{X2}(x,y)\pi_Y^X(y)dy + \int_x^T h_{X1}(x,y)\pi_Y^X(y)dy$
- Optimal release time for X: $x^* = \arg\max_{0\leq x\leq T} M_X(x)$
- Above arguments slightly modified in absence of risk neutrality, i.e., when considering a utility function u_X

  $g_{X1}(x,y) = A_X(x)\times u_X(p_X - c_X(x)) + [1 - A_X(x)]\times u_X(-(c_{1X}(x) + c_{2X}N_X(x)))$

  $$\begin{aligned} g_{X2}(x,y) \quad = \quad & A_Y(y) \times u_X(-[c_{1X}y + c_{2X}N_X(y)]) + [1 - A_Y(y)] \times \\ & \times \{A_X(x)u_X([p_X - c_X(x)]) + [1 - A_X(x)]\, u_X(-[c_{1X}x + c_{2X}N_X(x)])\} \end{aligned}$$

ADVERSARIAL SOFTWARE TESTING

- All the elements introduced above are standard in the decision analysis and software reliability literature and practice, except for those entailing strategic thinking:
  - $A_Y(y)$ (purchase probability of Y's software)
  - $\pi_Y^X(y)$ (X's beliefs about Y releasing its product at time y)
- Need for procedures to facilitate their assessment, starting with $\pi_Y^X(y)$
- Look at Y's perspective on product release
- Remember that Y has a cost function $c_Y(t)$ and a purchase probability function $A_Y(t)$ for a fixed price p_Y, with similar properties and definitions than those of X
- Presenting now an approach to obtain an estimate $\widehat{\pi}_Y^X(t)$ of $\pi_Y^X(t)$ reflecting upon the optimisation problem faced by Y

ADVERSARIAL SOFTWARE TESTING

- Suppose X has complete knowledge about Y's behaviour, i.e., c_{1Y}, c_{2Y}, c_{3Y}, p_Y, $\lambda_Y(t)$, $A_Y(t)$ and $\pi_X^Y(t)$ (which models Y's beliefs about X's release time)

- $\Rightarrow X$ could guess Y's actual optimal release time y^*, using the previous computations by interchanging X and Y

- But we have uncertainty about Y's elements so that we
  - model such uncertainty through probability measures $\Pi_X^Y(t)$, C_{1Y}, C_{2Y}, C_{3Y}, P_Y, $\mathcal{A}_Y$ and $\mathcal{N}_Y(t)$ over the space of suitable densities $\pi_X^Y(t)$, constants c_{1Y}, c_{2Y}, c_{3Y}, p_Y, functions A_Y and processes $N_Y(t)$, respectively
  - make a sufficiently large number of draws from these components, compute the corresponding optimal release time y^* for each draw, and estimate an empirical distribution over y^*, which will be considered as the estimate $\widehat{\pi}_Y^X(y)$
  - $\Rightarrow X$ will be able to compute its optimal release time x^*

ADVERSARIAL SOFTWARE TESTING

- The random ingredients could be specified gathering all information available and modelling with standard expert judgement

- Here we consider several heuristics based on adding some uncertainty to the judgements concerning X

- Y's random beliefs about X's decision $\Pi_X^Y(t)$
  - Transform the time interval $[0, T]$ into the unit interval via the transformation $t \to t/T$, $0 \leq t \leq T$
  - Consider suitable densities $\pi_X^Y(t)$ in the space of all beta densities over $[0, 1]$ or a proper subset, if X feels capable of adding some constraints about their parameters, e.g. by fixing lower and/or upper bounds over mean and/or variance of the beta distributions
  - Randomly generate densities from such class, e.g., drawing a uniform distribution over both parameters of the beta distribution or its mean-variance pair

ADVERSARIAL SOFTWARE TESTING

- Y's random beliefs about X's decision $\Pi_X^Y(t)$
  - Use distortion function as in Arias-Nicolas, Ruggeri and Suárez-Llorens (2016)
  - Start from an absolutely continuous (for simplicity) pdf $\pi_X(t)$ and its cdf $\Pi_X(t)$, expressing X's opinion on Y's release time and build a random space of cdf's $\pi_X^Y(t)$ around it
  - Consider distortion functions $h(t)$, i.e. non-decreasing functions such that $h : [0, 1] \to [0, 1]$, $h(0) = 0, h(1) = 1$
  - Apply $h(\cdot)$ to $\Pi_X(t)$ and obtain random pdf's $\Pi_{hX}^Y(t) = h(\Pi_X(t))$ and cdf's $\pi_{hX}^Y(t) = h^{'}(\Pi_X(t))\pi_X(t)$
  - Consider a band around $\Pi_X(t)$ taking one convex and one concave distortion function to get, respectively, its lower and upper bounds
  - A useful choice for a distortion function is $h(t) = t^\alpha$, which is convex for $0 < \alpha < 1$ and concave for $\alpha > 1$
  - Randomness is induced by, say, considering that α follows a uniform distribution on a certain interval

ADVERSARIAL SOFTWARE TESTING

- Uncertainty about Y's costs
  - Model X's uncertainty about c_{1Y}, c_{2Y} and c_{3Y} considering independent (Gaussian) distributions centered around the corresponding values c_{1X}, c_{2X}, c_{3X}
  - Alternatively, if X can provide upper and lower bounds for c_{1Y}, c_{2Y} and $d_Y = c_{3Y} - c_{2Y}$, then independent shifted beta distributions could be considered
  - The variances of those distributions will be determined by X depending on the confidence about the chosen means
- Uncertainty about Y's price P_Y
  - In absence of further information consider a (Gaussian) distribution with mean p_X and variance σ^2 denoting the degree of uncertainty around p_X
- Uncertainty about Y's purchase probability $A_Y(y)$
  - Transform $A_X(x) \rightarrow a\,[A_X(x)]^b$, with $a \in [0, 1]$ (decreasing effect) and $b \in [0, 1]$ (increasing effect)
  - a and b randomly generated to obtain values of $A_Y(y)$

ADVERSARIAL SOFTWARE TESTING

- Uncertainty about Y's fault discovery process $\mathcal{N}_Y(t)$
  - Suppose X has chosen a functional form for $N_X(t)$ and estimated its parameters and obtained an estimate $\tilde{m}_X(t)$ for its mean value function
  - First alternative: generate values of the parameters of $\mathcal{N}_Y(t)$ from distributions centered around X's estimated parameters (e.g. posterior distributions)
  - Second alternative: Bayesian non-parametric approach with mean value function as a random measure M, generated by a Gamma process, conjugate w.r.t. the Poisson process (Lo, 1982)
  - Gamma process centered around $\tilde{m}_X(t)$ so that at each interval $[t_0, t_1]$ the mean value function is generated by a Gamma distribution with mean $\tilde{m}_X(t_1) - \tilde{m}_X(t_0)$
  - The variance of the Gamma distribution could determine how close the fault discovery process $N_Y(t)$ is to $N_X(t)$
  - Further details can be found in Cavallo and Ruggeri (2001)

ADVERSARIAL SOFTWARE TESTING: EXAMPLE

- Example based on Zeephongsekul and Chiera (1995)
- Life cycle length $T = 2000$ days
- Cost parameters: $c_{1X} = 0.5$, $c_{2X} = 1$, $c_{3X} = 5$
- Selling price $p_X = 5000$
- Purchase probability $A_X(t) = 0.0002t(10 - 0.005t)$
- Fault discovery process $N_X(t)$: NHPP with mean value function $m_X(t) = at^c$ (power law process) and MLEs of parameters given by $\widehat{a} = 0.256$ and $\widehat{c} = 0.837$, from Zeephongsekul and Chiera (1995) and based on data from Okumoto (1979)
- Cost function with utility function u_X assumed to be the identity ($\Rightarrow$ Risk neutrality)

ADVERSARIAL SOFTWARE TESTING: EXAMPLE

- Cost parameters follow distributions centered around the c_X values:
  - $c_{1Y} \sim N(0.5, 0.02) = N(c_{1X}, 0.02)$
  - $c_{2Y} \sim N(1, 0.05) = N(c_{2X}, 0.05)$
  - $c_{3Y} \sim N(5, 0.5) = N(c_{3X}, 0.5)$
- Selling price $p_Y \sim N(5000, 250) = N(p_X, 250)$
- Random purchase probability $A_Y(t) \sim \tilde{d} A_X(t)^{\tilde{b}}$, with $\tilde{d} \sim U(0, 1)$ and $\tilde{b} \sim U(0, 1)$
- The random fault discovery process $N_Y(t)$ is a NHPP with random mean value function $m_Y(t) = \tilde{a} t^{\tilde{c}}$ with $\tilde{a} \sim N(0.256, 0.05)$ and $\tilde{c} \sim N(0.837, 0.05)$
- Beliefs of Y over X's release time t given by $t/T \sim \beta e(\alpha, \alpha)$, with $\alpha \sim U(1, 3)$
- Y's random cost function $c_Y(t) = c_{1Y} t + c_{2Y} N_Y(t) + c_{3Y} [N_Y(T) - N_Y(t)]$
- Deterministic utility function U_Y: identity $\Rightarrow$ risk neutrality

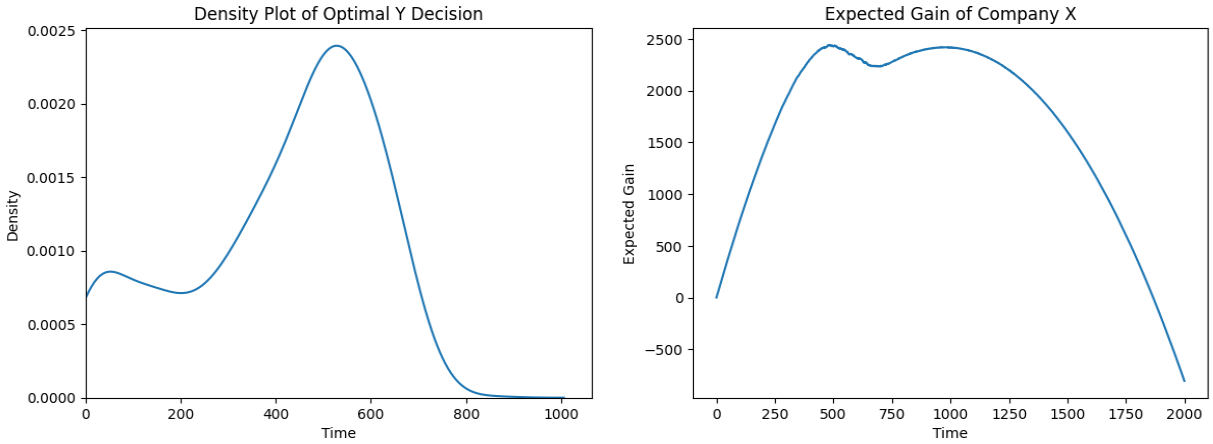
ADVERSARIAL SOFTWARE TESTING: EXAMPLE

- Forecasting Y's release decision
  - Maximise the objective function $M_Y(y) = \int h_Y(x,y)\pi_X^Y(x)dx$
  - For $i = 1, ..., K$
    - Sample c_{1Y}, c_{2Y}, c_{3Y}, p_Y, A_Y, N_Y, α (for π_X^Y, i.e. Y's beliefs on X's release)
    - Given the sampled α_i
      - generate a sample $z_j \sim \beta e(\alpha_i, \alpha_i)$, $j = 1, ..., N$
      - get $x_j = z_j \times T$, $j = 1, ..., N$
    - Monte Carlo approximation $M_Y^i(y)$ through $\frac{1}{N}\sum_{j=1}^{N} h_Y(x_j, y) = \frac{1}{N}[\sum_{x_j<y} h_{Y2}(x_j, y) + \sum_{y<x_j} h_{Y1}(x_j, y)] =$ (omitted)
    - $\Rightarrow$ find $y_i^* = \arg\max_{0\le x\le T} M^i{}_Y(y)$
  - $\Rightarrow$ Get approximate df $\widehat{\Pi}_Y^X(y) = card\{y_i^* : y_i^* \le y\}/K$

ADVERSARIAL SOFTWARE TESTING: EXAMPLE

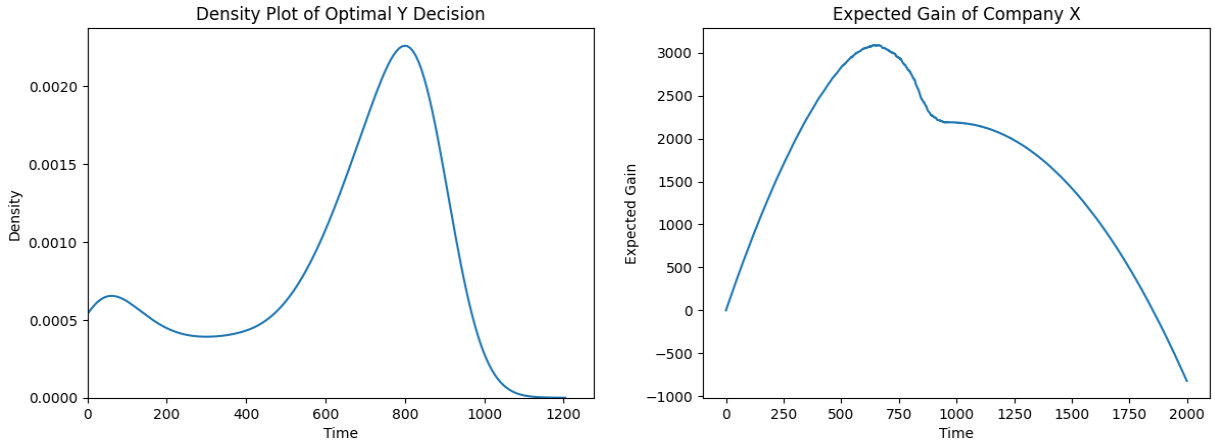
- Deciding X's optimal release
  - Find $x^* = \arg\max_{0 \leq x \leq T} M_X(x)$
  - Maximise the objective function $M_X(x) = \int h_X(x,y)\pi_Y^X(y)dy$
  - Approximate df $\widehat{\Pi}_Y^X(y) = card\{y_i^* : y_i^* \leq y\}/K$
  - Monte Carlo approximation through
    $\frac{1}{K}\sum_{i=1}^{K} h_X(x,y_i^*) = \frac{1}{K}[\sum_{y_i^* \leq x} h_{X2}(x,y_i^*) + \sum_{y_i^* \geq x} h_{X1}(x,y_i^*)]$ = (omitted)

ADVERSARIAL SOFTWARE TESTING: EXAMPLE



- $\beta e(\alpha, \alpha)$ distribution (mean 0.5) on X's release $\Rightarrow$ guess $1000 = 0.5 * 2000$
- LEFT: Y's optimal release time up to 800 days (out of 2000) with some incentive to very early release but the optimal ones are between 300 and 700
- RIGHT: bimodality in X's optimal release, with two possible strategies, one before Y's release and one after it
- X's optimal release occurs on day 483 for an expected gain of 2,442

ADVERSARIAL SOFTWARE TESTING: EXAMPLE



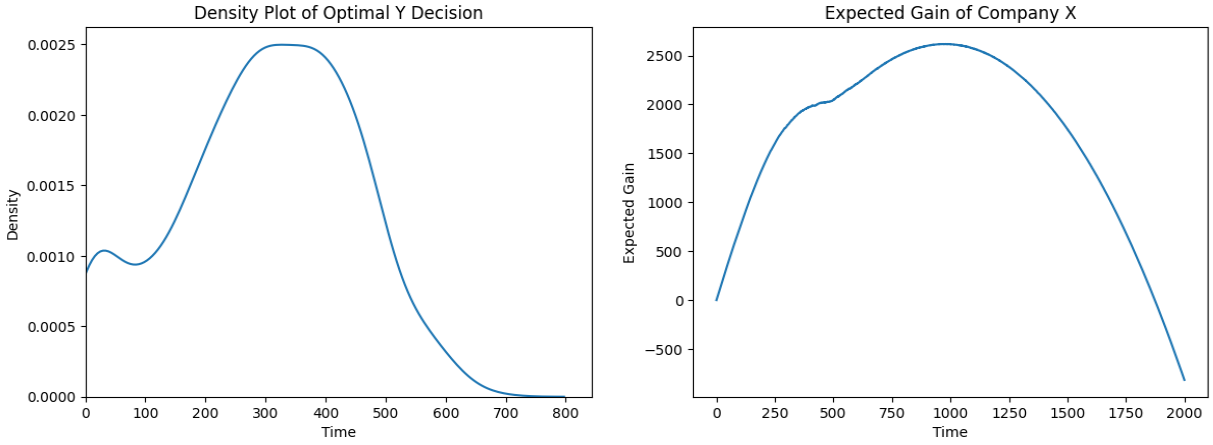
- X thinks that Y thinks that X will release later
  $\Rightarrow \beta e(\alpha, \alpha)$ on X's release replaced with $\beta e(3\alpha, \alpha) \Rightarrow$ guess $1,500 = 0.75{*}2000$

- LEFT: Y's optimal release up to 1200 days with some incentive to very early release and optimal ones between 700 and 900 (compare with 300 and 700)

- RIGHT: X's optimal release is before Y's one

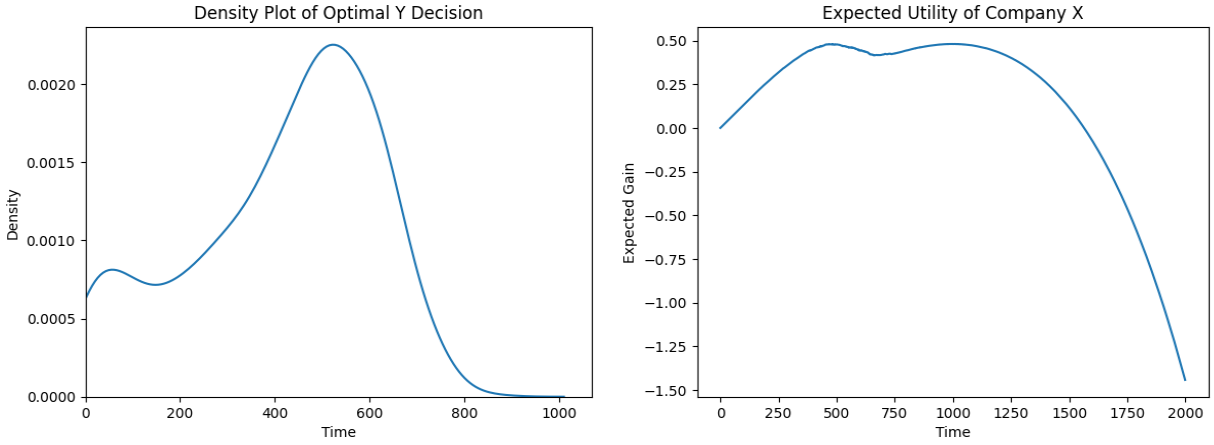
- X's optimal release on day 663 for an expected gain of 3,091 (earlier 483 and 2,442)

ADVERSARIAL SOFTWARE TESTING: EXAMPLE



- X thinks that Y thinks that X will release earlier
  $\Rightarrow \beta e(\alpha, \alpha)$ on X's release replaced with $\beta e(\alpha, 3\alpha) \Rightarrow$ guess $500 = 0.25 * 2000$
- LEFT: Y's optimal release up to 800 days with some incentive to very early release and high-risk early release between 200 and 500 (earlier 300 & 700 and 700 & 900)
- RIGHT: X's optimal release is well after the Y's high-risk one
- X's optimal release on day 978 with expected gain of 2,619 (earlier 483 & 2,442 and 663 & 3,091)

ADVERSARIAL SOFTWARE TESTING: EXAMPLE



- Risk averse $X \Rightarrow$ identity utility replaced with constant absolute risk averse (CARA) model given by $u(x) = 1 - \exp(-\rho x)$, with risk aversion parameter $\rho = 0.001$
- LEFT: Y's optimal release between 300 and 700 unchanged w.r.t. the first plot
- RIGHT: Still bimodal distribution for X's optimal release, but tendency to be more conservative and wait more
- X's optimal release on day 1003 (483 under identity) with expected utility (no more gain!) of 0.48

AST: CURRENT WORK

- Multiple producers
  - Instead of $x < y$ and $x > y$, consider order statistics and position X's release time between $x_{(i-1)}$ and $x_{(i+1)}$ for all i's
  - Similar formulas w.r.t. previous ones
- Multiple decision variables
  - So far the A_X purchase probability has been considered only as a function of the release time but it should depend also on other variables, like price and quality of the software
- Multiple buyers

REFERENCES

- Banks, D., Rios, J., and Rios Insua, D. (2015). *Adversarial Risk Analysis* (Vol. 343). CRC Press.

- Rios Insua, D., Rios, J. and Banks, D. (2009). Adversarial risk analysis. *Journal of the American Statistical Association*, 104, 841-854.

- Gonzalez-Ortega, J., Soyer, R., Rios Insua, D. and Ruggeri, F. (2021), An Adversarial Risk Analysis Framework for Batch Acceptance Problem. *Decision Analysis*, 18, 25-40.

- Rios Insua, D., Ruggeri, F., Soyer, R. and Rasines, D.G. (2018), Adversarial issues in reliability. *European Journal of Operational Research*, 266, 1113-1119.

- Rios Insua, D., Ruggeri, F., Soyer, R. and Wilson S. (2020), Advances in Bayesian Decision Making in Reliability. *European Journal of Operational Research*, 282, 1-18.

REFERENCES

- Gonzalez-Ortega, J., Rios Insua, D., Ruggeri, F. and Soyer, R. (2021), Hypothesis Testing in Presence of Adversaries. *The American Statistician*, 75, 31-40.

- Naveiro, R., Redondo, A., Rios Insua, D. and Ruggeri, F. (2019), Adversarial classification: An adversarial risk analysis approach. *International Journal of Approximate Reasoning*, 113, 133-148.

- Gallego, V., Naveiro, R., Redondo, A., Rios Insua, D. and Ruggeri, F., Protecting Classifiers From Attacks. *Statistical Science*, 39, 449-468.

- Soyer, R., Ruggeri, F., Rios Insua, D., Pierce, C. and Guevara, C., An adversarial risk analysis framework for software release decision support. To appear in *Risk Analysis*.

- Rios Insua, D., Ruggeri, F., Alfaro, C. and Gomez, J. (2016), Robustness for Adversarial Risk Analysis. In *Robustness Analysis in Decision Aiding, Optimization and Analytics*, M. Doumpos, C. Zopounidis and E. Grigoroudis Eds., Springer, 19-58.

REFERENCES

- Yang, J., Joshi, C. and Ruggeri, F. (2024), On Global Robustness of an Adversarial Risk Analysis Solution. *Statistica Neerlandica*, 78, 776-795.

- Arias, P., Ruggeri, F. and Suarez-Llorens, A. (2016), New classes of priors based on stochastic orders and distortion functions. *Bayesian Analysis*, 11, 1107-1136.

- Ruggeri, F., Sanchez-Sanchez, M., Sordo, M.A. and Suarez-Llorens, A. (2020), On a new class of multivariate prior distributions: theory and application in reliability. *Bayesian Analysis*, 16, 31-60.

- Rios Insua, D. and Ruggeri, F. Eds. (2000), *Robust Bayesian Analysis*, Springer, New York, USA.

- Cavallo, D. and Ruggeri, F. (2001), Bayesian models for failures in a gas network, *Safety and Reliability*, E. Zio, M. Demichela and N. Piccinini, Eds. , pp. 1963-1970, Politecnico di Torino Editore.